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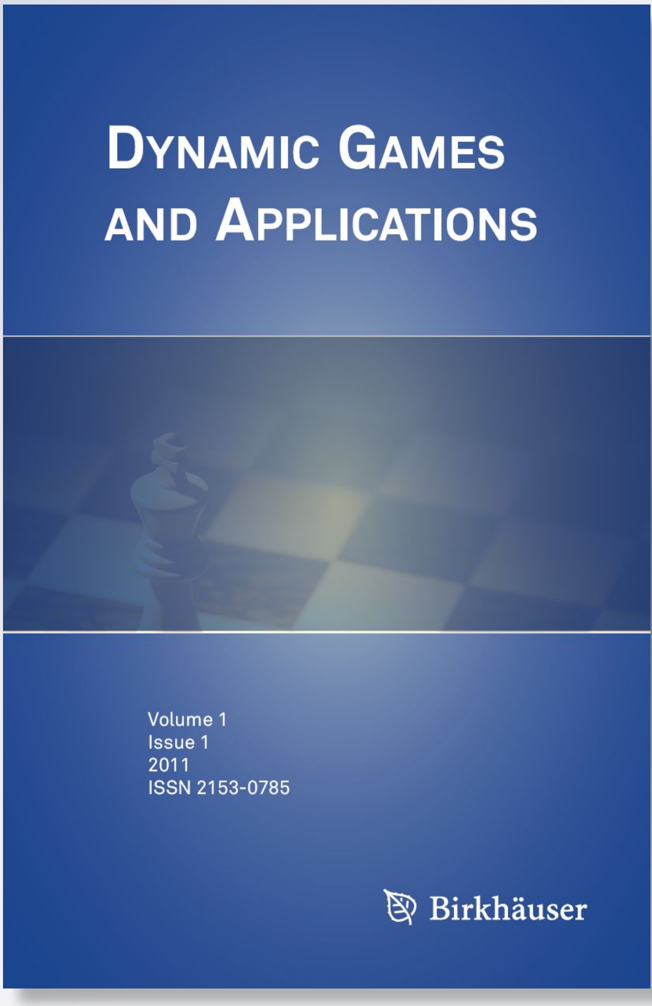

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The Envelope Theorem for Locally Differentiable Nash Equilibria of Discounted and Autonomous Infinite Horizon Differential Games

Chen Ling · Michael R. Caputo

Abstract The envelope theorem is extended to cover the class of discounted and autonomous infinite horizon differential games that possess locally differentiable Nash equilibria. The theorems cover open-loop and feedback information structures and are applied to an analytically solvable linear-quadratic game. The linear-quadratic structure permits additional insight into the theorems that is not available in the general case. With open-loop information, for example, the costate variable is shown to uniformly overstate the shadow value of the state variable, but the growth rates of the two are identical.

Keywords Envelope theorem · Differential games · Open-loop Nash equilibria · Feedback Nash equilibria

1 Introduction

The envelope theorem describes how the optimal value of the objective function of an optimization problem changes as the parameters of the problem change. As any first-year Ph.D. student in economics can attest, not only has it led to significant economic insights into single-player optimization problems, it also proved to be particularly useful in applied work. But because the envelope theorem is at present relatively underdeveloped for game-theoretic problems, it finds far less use in this setting. For example, the past several decades has witnessed an explosion of papers devoted to the derivation of game-theoretic equilibria in highly stylized games using simple functional forms, whereas the envelope theorem has received comparatively little attention in the extant game theory literature.

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There is some recent evidence that this is changing. For example, Caputo [3, 4] established an envelope theorem for a generic class of static games with locally differentiable Nash and Stackelberg equilibria, respectively. Caputo [6] further extended the envelope theorem to cover a general class of finite horizon differential games that possess locally differentiable Nash equilibria, with either open-loop or feedback information structures. Even more recently, Van Gorder and Caputo [16] derived an envelope theorem for a generic class of finite horizon differential games with an open-loop information structure in which there exist locally differentiable Stackelberg equilibria.

The present paper complements the above four by deriving the envelope theorem for the seemingly ubiquitous class of discounted and autonomous infinite horizon differential games possessing locally differentiable Nash equilibria. Two different information structures and their corresponding Nash equilibria are considered, to wit, (i) an *open-loop* information structure and its associated open-loop Nash equilibria (OLNE), and (ii) a *feedback* information structure and its corresponding feedback Nash equilibria (FNE). The central results demonstrate that if a certain transversality condition holds in the limit of the planning horizon, then the envelope expressions for the aforesaid class of games under consideration are essentially the same as those derived by [6].

Of all the envelope expressions, the change in an optimal value function with respect to a change in the initial value of a state variable is arguably the most discussed and important in economics. As may be seen in Theorems 9.1 and 14.10 of [5], the said envelope expression for a general class of optimal control problems is equal to the initial value of the corresponding costate variable. As a result, the initial value of the costate variable has the economic interpretation of the shadow value of the initial state variable in optimal control problems. Seeing as this envelope result and its resulting economic interpretation are so basic to intertemporal economic theory, the said interpretation has at times been naively extended to the setting of differential game theory without proof that it is indeed valid, as documented by [6]. One contribution of the present paper is the demonstration that Caputo's [6] envelope result for the initial state vector, namely, that the initial value of the costate vector is equal to the shadow value of the initial state vector for FNE, but not for OLNE, carries over to the class of differential games contemplated here.

Another contribution of the paper, and arguably the most important, is the application of the envelope theorem established here to an infinite horizon linear-quadratic differential game, the work-horse structure of the field. The linear-quadratic structure permits an analytical solution for the OLNE and FNE, the associated costate variables, and all the envelope expressions. As such, an explicit examination of the envelope results for a linear-quadratic game and a comparison of the initial value of a costate variable with the shadow value of the initial state variable are possible. Thus the linear-quadratic differential game permits the explicit demonstration of the aforementioned result that for OLNE, the initial value of the costate variable is not the shadow value of the initial state variable, but that it is for FNE.

Two surprising conclusions also emerge from the analysis of the linear-quadratic differential game, viz., (i) the absolute value of the costate variable is larger than the absolute value of the shadow value of the state variable for OLNE, but (ii) their growth rates are identical. These conclusions give some indication of how misleading it may be to treat a costate variable as the shadow value of the associated state variable for OLNE. Finally, the linear-quadratic game provides an opportunity to examine the magnitude of the difference between the shadow value of a state variable and its corresponding costate variable. To this end, a rudimentary numerical calculation is provided to give one a better feel for this difference in the linear-quadratic case.

We close this section by indicating the notational conventions adopted throughout the paper: (a) all vectors are column vectors and are indicated in boldface, (b) the derivative of a scalar-valued function with respect to a column vector is a row vector, (c) the derivative of a vector-valued function with respect to a vector is a Jacobian matrix, the number of rows of which equal the number of functions being differentiated, and the number of columns of which equal the number of elements in the vector that the derivative is taken with respect to, (d) gradient vectors and Jacobian matrices are denoted by boldface subscripts on the function being differentiated, and (e) the symbol "†" denotes transposition.

2 Definitions and Problem Statement

Consider the ubiquitous class of discounted infinite horizon differential games consisting of a finite number of P players, indexed by $p \in \{1, 2, \ldots, P\}$. Denote player-specific variables, functions, and parameters by upper indices. The state of the differential game at each instant $t \in [0, +\infty)$ is given by the vector $\mathbf{x}(t) \in \mathbb{R}^N$. The initial value of the state vector, denoted by $\mathbf{x}(0)$, is fixed at the value $\mathbf{x}_0 \in \mathbb{R}^N$. At each instant $t \in [0, +\infty)$, each player $p \in \{1, 2, \ldots, P\}$ chooses a control vector $\mathbf{u}^p(t) \in \mathbb{R}^{M^p}$ so as to maximize a payoff functional. Furthermore, define $\mathbf{u}^{-p}(t) \stackrel{\text{def}}{=} (\mathbf{u}^1(t), \mathbf{u}^2(t), \ldots, \mathbf{u}^{p-1}(t), \mathbf{u}^{p+1}(t), \ldots, \mathbf{u}^P(t))$ as the vector consisting of all the other players' control vectors at time t.

The state of the differential game evolves according to the system of ordinary differential equations $\dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}^1(t), \ldots, \mathbf{u}^P(t); \boldsymbol{\alpha})$, where $\boldsymbol{\alpha} \in \mathbb{R}^A$ is a vector of time independent parameters that enters the transition function $\mathbf{g}(\cdot)$ and the instantaneous payoff function $f^p(\cdot)$ of each player $p \in \{1, 2, \ldots, P\}$. The integrand of the objective functional of every player contains an exponential discount factor with a common and constant discount rate $r \in \mathbb{R}_{++}$. The differential game is otherwise autonomous, i.e., $f^p(\cdot)$, $p \in \{1, 2, \ldots, P\}$, and $\mathbf{g}(\cdot)$ do not explicitly depend on the time variable t.

Putting the above together, the generic class of autonomous and discounted infinite horizon differential games under consideration is therefore given by

$$W^p(\beta) \stackrel{\text{def}}{=} \max_{\mathbf{u}^p(\cdot)} \int_0^{+\infty} e^{-rt} f^p\big(\mathbf{x}(t), \mathbf{u}^p(t), \mathbf{u}^{-p}(t); \boldsymbol{\alpha}\big)\, dt$$
$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}\big(\mathbf{x}(t), \mathbf{u}^1(t), \ldots, \mathbf{u}^P(t); \boldsymbol{\alpha}\big), \qquad \mathbf{x}(0) = \mathbf{x}_0, \tag{1}$$

for $p \in \{1, 2, \ldots, P\}$, where $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, r, \mathbf{x}_0) \in \mathbb{R}^{A+1+N}$ is the vector of parameters of the game.

3 The Envelope Theorem for Open-Loop Nash Equilibria

The goal of this section is to derive the envelope theorem for the differential game defined by Eq. (1) under the assumption that the information structure of the game is open-loop. The following assumptions are therefore imposed on the game.

(A.1) $f^p(\cdot)$, $p \in \{1, 2, \ldots, P\}$, and $\mathbf{g}(\cdot)$ are $C^{(1)}$ on their domains, $\|f(\cdot)\| \leq C$, and $\|f_{\boldsymbol{\alpha}}(\cdot)\| \leq C$, where C is a positive constant and $\|\cdot\|$ is the Euclidean norm.

(A.2) The information structure for every player $p \in \{1, 2, \ldots, P\}$ is *open-loop*.

(A.3) An OLNE exists for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ; \delta)$, and is denoted by the P-tuple of vectors $\mathbf{v}(t; \boldsymbol{\beta}) = (\mathbf{v}^1(t; \boldsymbol{\beta}), \mathbf{v}^2(t; \boldsymbol{\beta}), \ldots, \mathbf{v}^P(t; \boldsymbol{\beta})) \in \mathbb{R}^{M^1} \times \mathbb{R}^{M^2} \times \cdots \times \mathbb{R}^{M^P}$, where $\mathbf{z}(t; \boldsymbol{\beta}) \in \mathbb{R}^N$ is the associated state path, $\boldsymbol{\lambda}^p(t; \boldsymbol{\beta}) \in \mathbb{R}^N$ is the corresponding current-value costate path of player $p \in \{1, 2, \ldots, P\}$, and $\boldsymbol{\lambda}(t; \boldsymbol{\beta}) \stackrel{\text{def}}{=} (\boldsymbol{\lambda}^1(t; \boldsymbol{\beta}), \boldsymbol{\lambda}^2(t; \boldsymbol{\beta}), \ldots, \boldsymbol{\lambda}^P(t; \boldsymbol{\beta})) \in \mathbb{R}^{NP}$.

(A.4) The vector-valued functions $(\mathbf{z}(\cdot), \mathbf{v}(\cdot), \boldsymbol{\lambda}(\cdot)) \in C^{(1)}$ for all $(t; \beta) \in [0, +\infty) \times B(\boldsymbol{\beta}^{\circ}; \delta)$.
(A.5) $\lim_{t\to+\infty} e^{-rt}\boldsymbol{\lambda}^{p}(t; \boldsymbol{\beta})^{\dagger}\partial\mathbf{z}(t; \boldsymbol{\beta})/\partial\boldsymbol{\beta} = \mathbf{0}^{\dagger}_{A+1+N}$ for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^{\circ}; \delta)$ and $p \in \{1, 2, \ldots, P\}$.

Assumption (A.1) guarantees that the objective functional of each player converges for all admissible pairs $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ due to the presence of the exponential discount factor in the integrand. Furthermore, as will be shown in the course of proving Theorem 1, it also implies that the optimal value function $W^{p}(\cdot)$ of every player belongs to the class of locally $C^{(1)}$ functions and that an interchange of the integration and differentiation operators is permitted, both of which are necessary for a differential characterization of the envelope theorem.

Supposition (A.2) specifies the information available to every player at each point in time in the game. In particular, an open-loop information structure implies that the corresponding OLNE at time t depends explicitly on t, the given initial value of the state vector $\mathbf{x}_0$, and the parameters $(\boldsymbol{\alpha}, r)$. Importantly, an OLNE at time t is not a function of the state vector at time t, namely, $\mathbf{x}(t)$. Indeed, the open-loop information structure is evident in the notation employed for an OLNE, namely, $\mathbf{v}(t; \boldsymbol{\alpha}, r, \mathbf{x}_0)$.

Assumption (A.3) presumes the existence of an OLNE. Alternatively, one can impose assumptions on $f^{p}(\cdot)$, $p \in \{1, 2, \ldots, P\}$, and $\mathbf{g}(\cdot)$ that imply the existence of an OLNE. However, imposing such assumptions on the differential game implies that the resulting envelope results are not intrinsic to it but are instead conditioned on the said sufficient conditions for existence. Consequently, such an approach reduces the generality and applicability of the results. Note that in order to keep the theorems as general as possible, we do not require that an OLNE converge to a steady state.

Supposition (A.4) is crucial to the analysis, as there is no guarantee that an OLNE is differentiable from assumption (A.1). Note, however, that this assumption precludes some classes of games, such as those that admit nonsmooth solutions, e.g., a bang–bang solution. It may therefore be stronger than needed to obtain the envelope results. Assumption (A.4) can be relaxed to allow a zero measure set of nonsmooth points for an OLNE, hence permitting the theorems to cover a broader class of games. Such a generalization, however, does not affect the envelope theorems nor the linear-quadratic game examined. As a result, the envelope theorems derived here are also valid for a slightly larger class of games than indicated.

Assumption (A.5) is unique to the infinite horizon class of differential games. It restricts the optimal values of the state and costate vectors in the limit of the planning horizon. The assumption plays essentially the same role as a transversality condition on the costate vector in finite horizon differential games, and hence acts as a transversality condition for the class of infinite horizon differential games under consideration. Note that the assumption only requires convergence along an equilibrium path, and thus is not all that restrictive in economics. Indeed, the assumption that an OLNE converges to a steady state, as is typically made in analyzing infinite horizon control problems and differential games, along with a regularity condition, are sufficient to yield assumption (A.5). To see this, first note that the aforementioned regularity condition requires that $\partial\mathbf{z}(t; \boldsymbol{\beta})/\partial\boldsymbol{\beta}$ is continuous on the closure of $B(\boldsymbol{\beta}^{\circ}; \delta)$, which implies that $\partial\mathbf{z}(t; \boldsymbol{\beta})/\partial\boldsymbol{\beta}$ is bounded. Then observe that the assumption that an OLNE converges to a steady state implies that $\lim_{t\to+\infty} \boldsymbol{\lambda}^{p}(t; \boldsymbol{\beta}) = \boldsymbol{\lambda}^{p}_{ss}(\boldsymbol{\alpha}, r)$, where $\boldsymbol{\lambda}^{p}_{ss}(\boldsymbol{\alpha}, r)$ is the steady state value of $\boldsymbol{\lambda}^{p}(t; \boldsymbol{\beta})$. Combining the previous two conclusions implies that $\lim_{t\to+\infty} e^{-rt}\boldsymbol{\lambda}^{p}(t; \boldsymbol{\beta})^{\dagger}\partial\mathbf{z}(t; \boldsymbol{\beta})/\partial\boldsymbol{\beta} = \mathbf{0}^{\dagger}_{A+1+N}$ owing to the exponential discount factor. In passing, note that because $\mathbf{z}(\cdot)$ is a $C^{(1)}$ function on $B(\boldsymbol{\beta}^{\circ}; \delta)$ by assumption (A.4), the continuity of $\partial \mathbf{z}(t; \boldsymbol{\beta})/\partial \boldsymbol{\beta}$ on the closure of $B(\boldsymbol{\beta}^\circ; \delta)$ is sufficient to reach the regularity condition. Intuitively, this condition means that there are no sudden jumps in the state variable when a parameter reaches the boundary of the set for which an OLNE exists.

By the definition of an OLNE, the optimal time-path of the pth player's control vector, i.e., $\mathbf{v}^p(t; \boldsymbol{\beta})$, is the solution to the optimal control problem

$$W^p(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{\mathbf{u}^p(\cdot)} \int_0^{+\infty} e^{-rt} f^p\big(\mathbf{x}(t), \mathbf{u}^p(t), \mathbf{v}^{-p}(t; \boldsymbol{\beta}); \boldsymbol{\alpha}\big)\, dt$$
$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}\big(\mathbf{x}(t), \mathbf{u}^p(t), \mathbf{v}^{-p}(t; \boldsymbol{\beta}); \boldsymbol{\alpha}\big), \qquad \mathbf{x}(0) = \mathbf{x}_0. \tag{2}$$

The *current-value Hamiltonian* function $H^p(\cdot)$ for optimal control problem (2) is defined as

$$H^p\big(\mathbf{x}, \mathbf{u}^p, \mathbf{v}^{-p}(t; \boldsymbol{\beta}), \boldsymbol{\lambda}^p; \boldsymbol{\alpha}\big) \stackrel{\text{def}}{=} f^p\big(\mathbf{x}, \mathbf{u}^p, \mathbf{v}^{-p}(t; \boldsymbol{\beta}); \boldsymbol{\alpha}\big) + \boldsymbol{\lambda}^{p\dagger} \mathbf{g}\big(\mathbf{x}, \mathbf{u}^p, \mathbf{v}^{-p}(t; \boldsymbol{\beta}); \boldsymbol{\alpha}\big). \tag{3}$$

As noted above, assumption (A.1) guarantees that the objective functional of each player converges for all admissible pairs. As a result, the necessary conditions obeyed by an OLNE are given by Theorem 6.11 of [1]. They take the ensuing form for the class of games under consideration:

$$\frac{\partial H^p}{\partial \mathbf{u}^p}\big(\mathbf{x}, \mathbf{u}^p, \mathbf{v}^{-p}(t; \boldsymbol{\beta}), \boldsymbol{\lambda}^p; \boldsymbol{\alpha}\big) = \mathbf{0}^\dagger_{M^p}, \tag{4}$$

$$\dot{\boldsymbol{\lambda}}^{p\dagger} = r\boldsymbol{\lambda}^{p\dagger} - \frac{\partial H^p}{\partial \mathbf{x}}\big(\mathbf{x}, \mathbf{u}^p, \mathbf{v}^{-p}(t; \boldsymbol{\beta}), \boldsymbol{\lambda}^p; \boldsymbol{\alpha}\big), \tag{5}$$

$$\dot{\mathbf{x}} = \mathbf{g}\big(\mathbf{x}, \mathbf{u}^p, \mathbf{v}^{-p}(t; \boldsymbol{\beta}); \boldsymbol{\alpha}\big), \qquad \mathbf{x}(0) = \mathbf{x}_0, \tag{6}$$

and hold for every $p \in \{1, 2, \ldots, P\}$. With Eqs. (4)–(6) in place, the central result of this section can be established.

Theorem 1 (Open-Loop Nash Equilibria) *If the differential game* (1) *satisfies assumptions* (A.1)–(A.5), *then for all* $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ; \delta)$ *and* $p \in \{1, 2, \ldots, P\}$, $W^p(\cdot) \in C^{(1)}$ *and*

$$\frac{\partial W^p}{\partial \boldsymbol{\alpha}}(\boldsymbol{\beta}) = \int_0^{+\infty} e^{-rt} \Bigg[\frac{\partial H^p}{\partial \boldsymbol{\alpha}}\big(\mathbf{z}(t; \boldsymbol{\beta}), \mathbf{v}(t; \boldsymbol{\beta}), \boldsymbol{\lambda}^p(t; \boldsymbol{\beta}); \boldsymbol{\alpha}\big) + \sum_{j=1, j\neq p}^{P} \frac{\partial H^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t; \boldsymbol{\beta}), \mathbf{v}(t; \boldsymbol{\beta}), \boldsymbol{\lambda}^p(t; \boldsymbol{\beta}); \boldsymbol{\alpha}\big) \frac{\partial \mathbf{v}^j}{\partial \boldsymbol{\alpha}}(t; \boldsymbol{\beta}) \Bigg] dt, \tag{7}$$

$$\frac{\partial W^p}{\partial \mathbf{x}_0}(\boldsymbol{\beta}) = \boldsymbol{\lambda}^p(0; \boldsymbol{\beta})^\dagger + \int_0^{+\infty} e^{-rt} \Bigg[\sum_{j=1, j\neq p}^{P} \frac{\partial H^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t; \boldsymbol{\beta}), \mathbf{v}(t; \boldsymbol{\beta}), \boldsymbol{\lambda}^p(t; \boldsymbol{\beta}); \boldsymbol{\alpha}\big) \frac{\partial \mathbf{v}^j}{\partial \mathbf{x}_0}(t; \boldsymbol{\beta}) \Bigg] dt, \tag{8}$$

$$\frac{\partial W^p}{\partial r}(\boldsymbol{\beta}) = \int_0^{+\infty} e^{-rt} \Bigg[-t f^p\big(\mathbf{z}(t; \boldsymbol{\beta}), \mathbf{v}(t; \boldsymbol{\beta}); \boldsymbol{\alpha}\big) + \sum_{j=1, j\neq p}^{P} \frac{\partial H^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t; \boldsymbol{\beta}), \mathbf{v}(t; \boldsymbol{\beta}), \boldsymbol{\lambda}^p(t; \boldsymbol{\beta}); \boldsymbol{\alpha}\big) \frac{\partial \mathbf{v}^j}{\partial r}(t; \boldsymbol{\beta}) \Bigg] dt. \tag{9}$$

Proof Given assumption (A.3), the optimal value function $W^p(\cdot)$ defined in Eq. (2) can be equivalently defined as

$$W^p(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \int_0^{+\infty} e^{-rt} f^p\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}\big)\, dt. \tag{10}$$

Recall that assumption (A.1) implies that the payoff functional of player p converges for all admissible pairs. Furthermore, the Weierstrass M-test and Theorem A.14.1 in [5] imply that $W^p(\cdot) \in C^{(1)}$ and

$$\frac{\partial}{\partial \boldsymbol{\beta}} \int_0^{+\infty} e^{-rt} f^p\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}\big)\, dt = \int_0^{+\infty} \frac{\partial}{\partial \boldsymbol{\beta}} e^{-rt} f^p\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}\big)\, dt \tag{11}$$

for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ; \delta)$ and $p \in \{1, 2, \ldots, P\}$. To prove Eq. (7), begin by differentiating Eq. (10) with respect to $\boldsymbol{\alpha}$ using Leibniz's rule to arrive at the expression

$$\begin{aligned} \frac{\partial W^p}{\partial \boldsymbol{\alpha}}(\boldsymbol{\beta}) = \int_0^{+\infty} e^{-rt} \Bigg[& \frac{\partial f^p}{\partial \mathbf{x}}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}\big) \frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) \\ & + \sum_{j=1}^{P} \frac{\partial f^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}\big) \frac{\partial \mathbf{v}^j}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) + \frac{\partial f^p}{\partial \boldsymbol{\alpha}}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}\big) \Bigg] dt. \end{aligned} \tag{12}$$

Next, differentiate the identity $\mathbf{g}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) - \dot{\mathbf{z}}(t;\boldsymbol{\beta}) \equiv \mathbf{0}_N$ with respect to $\boldsymbol{\alpha}$ to get

$$\begin{aligned} & \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}\big) \frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) + \sum_{j=1}^{P} \frac{\partial \mathbf{g}}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}\big) \frac{\partial \mathbf{v}^j}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) \\ & + \frac{\partial \mathbf{g}}{\partial \boldsymbol{\alpha}}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}\big) - \frac{\partial \dot{\mathbf{z}}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) = \mathbf{0}_{N \times A}. \end{aligned} \tag{13}$$

Now premultiply Eq. (13) by the $1 \times N$ row vector $e^{-rt} \boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger$, integrate the resulting inner product over $[0, +\infty)$, add the result to Eq. (12), and then use the definition of the Hamiltonian given in Eq. (3) to simplify the notation. These steps yield

$$\begin{aligned} \frac{\partial W^p}{\partial \boldsymbol{\alpha}}(\boldsymbol{\beta}) = \int_0^{+\infty} e^{-rt} \Bigg[& \frac{\partial H^p}{\partial \mathbf{x}}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \boldsymbol{\lambda}^p(t;\boldsymbol{\beta}); \boldsymbol{\alpha}\big) \frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) \\ & + \sum_{j=1}^{P} \frac{\partial H^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \boldsymbol{\lambda}^p(t;\boldsymbol{\beta}); \boldsymbol{\alpha}\big) \frac{\partial \mathbf{v}^j}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) \\ & + \frac{\partial H^p}{\partial \boldsymbol{\alpha}}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \boldsymbol{\lambda}^p(t;\boldsymbol{\beta}); \boldsymbol{\alpha}\big) - \boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \dot{\mathbf{z}}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) \Bigg] dt. \end{aligned} \tag{14}$$

Next, substitute the identity form of Eqs. (4) and (5) in Eq. (14) and then make use of

$$\begin{aligned} \frac{d}{dt}\left[e^{-rt} \boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) \right] = {} & e^{-rt} \boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \dot{\mathbf{z}}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) \\ & + e^{-rt} \dot{\boldsymbol{\lambda}}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) - r e^{-rt} \boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) \end{aligned} \tag{15}$$

to reduce Eq. (14) to

$$\frac{\partial W^p}{\partial \boldsymbol{\alpha}}(\boldsymbol{\beta}) = \int_0^{+\infty} e^{-rt}\Bigg[\frac{\partial H^p}{\partial \boldsymbol{\alpha}}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta}),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)\,dt$$
$$+\sum_{j=1,j\neq p}^{P}\frac{\partial H^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta}),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)\frac{\partial \mathbf{v}^j}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta})\Bigg]\,dt$$
$$-\lim_{t\to+\infty} e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) + \boldsymbol{\lambda}^p(0;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(0;\boldsymbol{\beta}). \tag{16}$$

Finally, using assumption (A.5) and the fact that $\mathbf{z}(0;\boldsymbol{\beta}) \equiv \mathbf{x}_0$ implies $\partial \mathbf{z}(0;\boldsymbol{\beta})/\partial\boldsymbol{\alpha} \equiv \mathbf{0}_{N\times A}$ in Eq. (16) reduces it to Eq. (7).

To prove Eq. (8), first differentiate Eq. (10) with respect to $\mathbf{x}_0$ using Leibniz's rule to get

$$\frac{\partial W^p}{\partial \mathbf{x}_0}(\boldsymbol{\beta}) = \int_0^{+\infty} e^{-rt}\Bigg[\frac{\partial f^p}{\partial \mathbf{x}}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)\frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta})$$
$$+\sum_{j=1}^{P}\frac{\partial f^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)\frac{\partial \mathbf{v}^j}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta})\Bigg]\,dt. \tag{17}$$

Then differentiate the identity $\mathbf{g}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha}) - \dot{\mathbf{z}}(t;\boldsymbol{\beta}) \equiv \mathbf{0}_N$ with respect to $\mathbf{x}_0$ to obtain

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)\frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta})$$
$$+\sum_{j=1}^{P}\frac{\partial \mathbf{g}}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)\frac{\partial \mathbf{v}^j}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta}) - \frac{\partial \dot{\mathbf{z}}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta}) = \mathbf{0}_{N\times N}. \tag{18}$$

Next, premultiply Eq. (18) by $e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger$, integrate the result over $[0,+\infty)$, add it to Eq. (17), and then simplify the resulting expression using Eqs. (3)–(5). These steps yield the expression

$$\frac{\partial W^p}{\partial \mathbf{x}_0}(\boldsymbol{\beta}) = \int_0^{+\infty} e^{-rt}\Bigg[-\dot{\boldsymbol{\lambda}}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta}) - \boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \dot{\mathbf{z}}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta}) + r\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta})$$
$$+\sum_{j=1,j\neq p}^{P}\frac{\partial H^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta}),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)\frac{\partial \mathbf{v}^j}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta})\Bigg]\,dt. \tag{19}$$

We now take a slightly different approach in finishing this proof than in proving Eq. (7). Specifically, integrate the first term of the integrand in Eq. (19) by parts to get

$$-\int_0^{+\infty} e^{-rt}\dot{\boldsymbol{\lambda}}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta})\,dt$$
$$= -\lim_{t\to+\infty} e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta}) + \boldsymbol{\lambda}^p(0;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(0;\boldsymbol{\beta})$$
$$+\int_0^{+\infty} e^{-rt}\Bigg[\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \dot{\mathbf{z}}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta}) - r\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta})\Bigg]\,dt. \tag{20}$$

Next, use assumption (A.5) and the fact that $\mathbf{z}(0;\boldsymbol{\beta}) \equiv \mathbf{x}_0$ implies $\partial \mathbf{z}(0;\boldsymbol{\beta})/\partial \mathbf{x}_0 \equiv \mathbf{I}_N$ to simplify Eq. (20), and then substitute the latter in Eq. (19) to arrive at Eq. (8).

To prove Eq. (9), first differentiate Eq. (10) with respect to r using Leibniz's rule to get

$$\frac{\partial W^p}{\partial r}(\boldsymbol{\beta}) = \int_0^{+\infty} e^{-rt}\Bigg[-tf^p\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big) + \frac{\partial f^p}{\partial \mathbf{x}}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)\frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta}) + \sum_{j=1}^{P}\frac{\partial f^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)\frac{\partial \mathbf{v}^j}{\partial r}(t;\boldsymbol{\beta})\Bigg]dt. \tag{21}$$

Next, differentiate the identity $\mathbf{g}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha}) - \dot{\mathbf{z}}(t;\boldsymbol{\beta}) \equiv \mathbf{0}_N$ with respect to r to obtain

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)\frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta}) + \sum_{j=1}^{P}\frac{\partial \mathbf{g}}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)\frac{\partial \mathbf{v}^j}{\partial r}(t;\boldsymbol{\beta}) - \frac{\partial \dot{\mathbf{z}}}{\partial r}(t;\boldsymbol{\beta}) = \mathbf{0}_N. \tag{22}$$

As before, premultiply Eq. (22) by $e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger$, integrate the result over $[0,+\infty)$, add it to Eq. (21), and then simplify the resulting expression using Eqs. (3)–(5) and

$$\begin{aligned}-\int_0^{+\infty} e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \dot{\mathbf{z}}}{\partial r}(t;\boldsymbol{\beta})\,dt &= -\lim_{t\to+\infty} e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta}) + \boldsymbol{\lambda}^p(0;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial r}(0;\boldsymbol{\beta}) \\ &\quad + \int_0^{+\infty} e^{-rt}\left[\dot{\boldsymbol{\lambda}}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta}) - r\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta})\right]dt\end{aligned} \tag{23}$$

to get the expression

$$\begin{aligned}\frac{\partial W^p}{\partial r}(\boldsymbol{\beta}) &= \int_0^{+\infty} e^{-rt}\Bigg[-tf^p\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big) \\ &\quad + \sum_{j=1,j\neq p}^{P}\frac{\partial H^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta}),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)\frac{\partial \mathbf{v}^j}{\partial r}(t;\boldsymbol{\beta})\Bigg]dt \\ &\quad - \lim_{t\to+\infty} e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta}) + \boldsymbol{\lambda}^p(0;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial r}(0;\boldsymbol{\beta}).\end{aligned} \tag{24}$$

Finally, using assumption (A.5) and the fact that $\mathbf{z}(0;\boldsymbol{\beta}) \equiv \mathbf{x}_0$ implies $\partial \mathbf{z}(0;\boldsymbol{\beta})/\partial r \equiv \mathbf{0}_N$, Eq. (24) reduces to Eq. (9). □

Theorem 1 shows that the envelope theorem for discounted and autonomous infinite horizon differential games with an open-loop information structure consists of two distinct effects when exogenous parameters change, a result fully parallel to that derived by [6, Theorem 2] for the generic class of nonautonomous finite horizon differential games with an open-loop information structure. Specifically, the envelope expressions consist of a *direct* or *explicit* effect given by the first term on the right-hand side of Eqs. (7)–(9), and a *strate-*

gic effect, which is captured by the second term on the right-hand side. The direct effect is identical to its counterpart in an optimal control problem—see [5, Theorem 14.10]—and captures the effect of a parameter change on the optimal value function of player p due to the explicit appearance of the parameter in the control problem of player p. The strategic effect, on the other hand, captures the impact of a parameter change on the optimal value function of player p by way of the response of the other $P-1$ players' optimal controls to the said parameter change. The strategic effect is identically zero in optimal control theory in view of the fact that there is only one player.

The envelope result in Eq. (8) indicates that the shadow value of the initial state vector, to wit, $\partial W^p(\boldsymbol{\beta})/\partial \mathbf{x}_0$, is not equal to the costate vector evaluated at the initial time, namely, $\boldsymbol{\lambda}^p(0;\boldsymbol{\beta})$. The difference between the two expressions is the aforementioned strategic effect. This implies that interpreting the initial value of the costate vector as the shadow value of the initial state vector in a differential game with an open-loop information structure is not, in general, correct. Moreover, given the difficulty of deriving closed-form solutions of differential games, Eq. (8) suggests that it will often times be difficult to determine the shadow value of the initial state vector in differential games with an open-loop information structure. Without putting any additional structure on the game, therefore, one cannot further compare the costate vector and the shadow value of the initial state vector.

In light of the preceding conclusion, Sect. 5 therefore contemplates an analytically solvable class of linear-quadratic differential games and computes its OLNE. The added structure permits surprising conclusions regarding the shadow value of the state variable and its corresponding costate variable that are not available in the general case. Another matter addressed in Sect. 5 is the magnitude of the difference between the costate variable and the shadow value of the state variable. In the meantime, however, we take up the envelope theorem for FNE.

4 The Envelope Theorem for Feedback Nash Equilibria

The analog to Theorem 1 for a feedback information structure is now established. Assumption (A.1) is retained, while assumptions (A.2)–(A.5) are replaced with their feedback counterparts.

(B.2) The information structure for every player $p \in \{1, 2, \ldots, P\}$ is *feedback*.

(B.3) A FNE exists for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$, and is denoted by the P-tuple of vectors $\mathbf{v}(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r) \stackrel{\text{def}}{=} (\mathbf{v}^1(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r), \mathbf{v}^2(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r), \ldots, \mathbf{v}^P(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r))$, where $\mathbf{z}(t;\boldsymbol{\beta}) \in \mathbb{R}^N$ is the associated state path, $\boldsymbol{\lambda}^p(t;\boldsymbol{\beta}) \in \mathbb{R}^N$ is the corresponding current-value costate path of player $p \in \{1, 2, \ldots, P\}$, and $\boldsymbol{\lambda}(t;\boldsymbol{\beta}) \stackrel{\text{def}}{=} (\boldsymbol{\lambda}^1(t;\boldsymbol{\beta}), \boldsymbol{\lambda}^2(t;\boldsymbol{\beta}), \ldots, \boldsymbol{\lambda}^P(t;\boldsymbol{\beta})) \in \mathbb{R}^{NP}$.

(B.4) The vector-valued functions $(\mathbf{z}(\cdot), \boldsymbol{\lambda}(\cdot)) \in C^{(1)}$ for all $(t;\boldsymbol{\beta}) \in [0,+\infty) \times B(\boldsymbol{\beta}^\circ;\delta)$ and $\mathbf{v}(\cdot) \in C^{(1)}$ for all $(t,\mathbf{x};\boldsymbol{\alpha},r) \in [0,+\infty) \times \{(\mathbf{x};\boldsymbol{\alpha},r) : (\mathbf{x};\boldsymbol{\alpha},r) = (\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r), \boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)\}$.

(B.5) $\lim_{t\to+\infty} e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger \partial\mathbf{z}(t;\boldsymbol{\beta})/\partial\boldsymbol{\beta} = \mathbf{0}^\dagger_{A+1+N}$ for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$ and $p \in \{1, 2, \ldots, P\}$.

As before, assumption (B.2) describes the information that is available to every player at each point in time of the game. In particular, the feedback information structure implies that a FNE at time t depends, in general, explicitly on t, the value of the state vector at time t, namely $\mathbf{x}(t)$, and the parameter vector $(\boldsymbol{\alpha}, r)$, as is clear from the notation $\mathbf{v}(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r)$ used to denote a FNE. A FNE also depends on the initial state, but such dependence is not direct, rather, it is indirect via the equilibrium state path $\mathbf{z}(t;\boldsymbol{\beta})$, as $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, r, \mathbf{x}_0)$. Furthermore, note that it is common to restrict the analysis of discounted and autonomous infinite horizon differential games to stationary FNE, i.e., FNE that do not depend explicitly on t. Nonetheless, it is entirely possible that the said class of games possess nonstationary FNE as well, i.e., FNE that depends explicitly on t, as noted by [10, p. 97 and Exercise 4.5]. Accordingly, we have allowed for nonstationary FNE in our assumptions and notation. Supposition (B.3) asserts the existence of a FNE and its corresponding state and costate paths, while assumption (B.4) presumes that the FNE is locally differentiable with respect to the parameters. Supposition (B.5) is analogous to (A.5), and is essentially an infinite horizon transversality condition. Note, in passing, the slight abuse of notation in moving from the open-loop to the feedback information structure.

By the definition of a FNE, the optimal time-path of the pth player's control vector, i.e., $\mathbf{v}^p(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r)$, is a solution to the optimal control problem

$$W^p(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{\mathbf{u}^p(\cdot)} \int_0^{+\infty} e^{-rt} f^p\big(\mathbf{x}(t), \mathbf{u}^p(t), \mathbf{v}^{-p}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big)\, dt$$

$$\text{s.t.}\quad \dot{\mathbf{x}}(t) = \mathbf{g}\big(\mathbf{x}(t), \mathbf{u}^p(t), \mathbf{v}^{-p}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big), \qquad \mathbf{x}(0) = \mathbf{x}_0. \tag{25}$$

The current-value Hamiltonian for player p is defined as

$$H^p\big(\mathbf{x}, \mathbf{u}^p, \mathbf{v}^{-p}(t, \mathbf{x}; \boldsymbol{\alpha}, r), \boldsymbol{\lambda}^p; \boldsymbol{\alpha}\big)$$
$$\stackrel{\text{def}}{=} f^p\big(\mathbf{x}, \mathbf{u}^p, \mathbf{v}^{-p}(t, \mathbf{x}; \boldsymbol{\alpha}, r); \boldsymbol{\alpha}\big) + \boldsymbol{\lambda}^{p\dagger}\mathbf{g}\big(\mathbf{x}, \mathbf{u}^p, \mathbf{v}^{-p}(t, \mathbf{x}; \boldsymbol{\alpha}, r); \boldsymbol{\alpha}\big). \tag{26}$$

Because assumption (A.1) guarantees that the objective functional of each player converges for all admissible pairs, Theorem 2.2 of [14] provides the necessary conditions that a FNE must satisfy, namely

$$\frac{\partial H^p}{\partial \mathbf{u}^p}\big(\mathbf{x}, \mathbf{u}^p, \mathbf{v}^{-p}(t, \mathbf{x}; \boldsymbol{\alpha}, r), \boldsymbol{\lambda}^p; \boldsymbol{\alpha}\big) = \mathbf{0}^\dagger_{M^p}, \tag{27}$$

$$\dot{\boldsymbol{\lambda}}^{p\dagger} = r\boldsymbol{\lambda}^{p\dagger} - \frac{\partial H^p}{\partial \mathbf{x}}\big(\mathbf{x}, \mathbf{u}^p, \mathbf{v}^{-p}(t, \mathbf{x}; \boldsymbol{\alpha}, r), \boldsymbol{\lambda}^p; \boldsymbol{\alpha}\big)$$
$$- \sum_{j=1, j\neq p}^{P} \frac{\partial H^p}{\partial \mathbf{u}^j}\big(\mathbf{x}, \mathbf{u}^p, \mathbf{v}^{-p}(t, \mathbf{x}; \boldsymbol{\alpha}, r), \boldsymbol{\lambda}^p; \boldsymbol{\alpha}\big)\frac{\partial \mathbf{v}^j}{\partial \mathbf{x}}(t, \mathbf{x}; \boldsymbol{\alpha}, r), \tag{28}$$

$$\dot{\mathbf{x}} = \mathbf{g}\big(\mathbf{x}, \mathbf{u}^p, \mathbf{v}^{-p}(t, \mathbf{x}; \boldsymbol{\alpha}, r); \boldsymbol{\alpha}\big), \qquad \mathbf{x}(0) = \mathbf{x}_0, \tag{29}$$

which hold for every $p \in \{1, 2, \ldots, P\}$. With these conditions in place, the ensuing envelope result for FNE may be proven.

Theorem 2 (Feedback Nash Equilibria) *If the differential game* (1) *satisfies assumptions* (A.1) *and* (B.2)–(B.5), *then for all* $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ; \delta)$ *and* $p = 1, 2, \ldots, P$, $W^p(\cdot) \in C^{(1)}$ *and*

$$\frac{\partial W^p}{\partial \boldsymbol{\alpha}}(\boldsymbol{\beta}) = \int_0^{+\infty} e^{-rt}\Bigg[\frac{\partial H^p}{\partial \boldsymbol{\alpha}}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big), \boldsymbol{\lambda}^p(t;\boldsymbol{\beta}); \boldsymbol{\alpha}\big)$$
$$+ \sum_{j=1, j\neq p}^{P} \frac{\partial H^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big), \boldsymbol{\lambda}^p(t;\boldsymbol{\beta}); \boldsymbol{\alpha}\big)\frac{\partial \mathbf{v}^j}{\partial \boldsymbol{\alpha}}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big)\Bigg]\, dt, \tag{30}$$

$$\frac{\partial W^p}{\partial \mathbf{x}_0}(\boldsymbol{\beta}) = \boldsymbol{\lambda}^p(0; \boldsymbol{\beta}), \tag{31}$$

$$\frac{\partial W^p}{\partial r}(\boldsymbol{\beta}) = \int_0^{+\infty} e^{-rt}\Bigg[-tf^p\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big) + \sum_{j=1, j\neq p}^{P} \frac{\partial H^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big), \boldsymbol{\lambda}^p(t;\boldsymbol{\beta}); \boldsymbol{\alpha}\big)\frac{\partial \mathbf{v}^j}{\partial r}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big)\Bigg]dt. \tag{32}$$

Proof By assumption (B.3), the optimal value function $W^p(\cdot)$ defined in Eq. (25) can be equivalently defined as

$$W^p(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \int_0^{+\infty} e^{-rt} f^p\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big)\, dt. \tag{33}$$

As before, assumption (A.1) implies that the payoff functional of player p converges for all admissible pairs. In addition, by the Weierstrass M-test and Theorem A.14.1 in [5] $W^p(\cdot) \in C^{(1)}$ and

$$\frac{\partial}{\partial \boldsymbol{\beta}} \int_0^{+\infty} e^{-rt} f^p\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big)\, dt = \int_0^{+\infty} \frac{\partial}{\partial \boldsymbol{\beta}} e^{-rt} f^p\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big)\, dt \tag{34}$$

for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ; \delta)$ and $p \in \{1, 2, \ldots, P\}$. To establish Eq. (30), first differentiate Eq. (33) with respect to $\boldsymbol{\alpha}$ using Leibniz's rule to get

$$\frac{\partial W^p}{\partial \boldsymbol{\alpha}}(\boldsymbol{\alpha}) = \int_0^{+\infty} e^{-rt}\Bigg[\frac{\partial f^p}{\partial \mathbf{x}}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big)\frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) + \sum_{j=1}^{P} \frac{\partial f^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big) \times \left[\frac{\partial \mathbf{v}^j}{\partial \mathbf{x}}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big)\frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) + \frac{\partial \mathbf{v}^j}{\partial \boldsymbol{\alpha}}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big)\right] + \frac{\partial f^p}{\partial \boldsymbol{\alpha}}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big)\Bigg]dt. \tag{35}$$

Next, differentiate the identity form of Eq. (29) with respect to $\boldsymbol{\alpha}$ to arrive at

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big)\frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) + \sum_{j=1}^{P} \frac{\partial \mathbf{g}}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big)$$

$$\times\left[\frac{\partial \mathbf{v}^j}{\partial \mathbf{x}}\big(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r\big)\frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta})+\frac{\partial \mathbf{v}^j}{\partial \boldsymbol{\alpha}}\big(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r\big)\right]$$
$$+\frac{\partial \mathbf{g}}{\partial \boldsymbol{\alpha}}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}\big(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r\big);\boldsymbol{\alpha}\big)-\frac{\partial \dot{\mathbf{z}}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta})\equiv \mathbf{0}_{N\times A}. \tag{36}$$

Now premultiply Eq. (36) by the $1\times N$ row vector $e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger$, integrate the resulting inner product over $[0,+\infty)$, add the result to Eq. (35), and then use the definition of the Hamiltonian given in Eq. (26) to simplify the notation. The result of these steps is the expression

$$\begin{aligned}\frac{\partial W^p}{\partial \boldsymbol{\alpha}}(\boldsymbol{\beta})=\int_0^{+\infty} e^{-rt}\Bigg[&\frac{\partial H^p}{\partial \mathbf{x}}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}\big(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r\big),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)\frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta})\\
&+\sum_{j=1}^{P}\frac{\partial H^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}\big(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r\big),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)\\
&\times\left[\frac{\partial \mathbf{v}^j}{\partial \mathbf{x}}\big(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r\big)\frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta})\right]\\
&+\sum_{j=1}^{P}\frac{\partial H^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}\big(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r\big),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)\frac{\partial \mathbf{v}^j}{\partial \boldsymbol{\alpha}}\big(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r\big)\\
&+\frac{\partial H^p}{\partial \boldsymbol{\alpha}}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}\big(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r\big),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)-\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \dot{\mathbf{z}}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta})\Bigg]dt.\end{aligned} \tag{37}$$

Next, substitute the identity form of Eqs. (27) and (28) in Eq. (37) to obtain

$$\begin{aligned}\frac{\partial W^p}{\partial \boldsymbol{\alpha}}(\boldsymbol{\beta})=\int_0^{+\infty} e^{-rt}\Bigg[&\frac{\partial H^p}{\partial \boldsymbol{\alpha}}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}\big(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r\big),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)\\
&+\sum_{j=1,j\neq p}^{P}\frac{\partial H^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}\big(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r\big),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)\frac{\partial \mathbf{v}^j}{\partial \boldsymbol{\alpha}}\big(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r\big)\\
&+r\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta})-\dot{\boldsymbol{\lambda}}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta})-\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \dot{\mathbf{z}}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta})\Bigg]dt.\end{aligned} \tag{38}$$

Then substitute

$$\begin{aligned}\frac{d}{dt}\left[e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta})\right]&=e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \dot{\mathbf{z}}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta})\\
&\quad+e^{-rt}\dot{\boldsymbol{\lambda}}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta})-re^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta})\end{aligned} \tag{39}$$

in Eq. (38) and employ assumption (B.5) and the fact that $\mathbf{z}(0;\boldsymbol{\beta})\equiv \mathbf{x}_0$ implies $\partial\mathbf{z}(0;\boldsymbol{\beta})/\partial\boldsymbol{\alpha}\equiv \mathbf{0}_{N\times A}$ in the resulting expression to derive Eq. (30).

To prove Eq. (31), first differentiate Eq. (33) with respect to $\mathbf{x}_0$ using Leibniz's rule to get

$$\frac{\partial W^p}{\partial \mathbf{x}_0}(\boldsymbol{\alpha}) = \int_0^{+\infty} e^{-rt}\Bigg[\frac{\partial f^p}{\partial \mathbf{x}}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big)\frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta}) + \sum_{j=1}^{P}\frac{\partial f^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big)\frac{\partial \mathbf{v}^j}{\partial \mathbf{x}}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big)\frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta})\Bigg]dt. \tag{40}$$

Then differentiate the identity form of Eq. (29) with respect to $\mathbf{x}_0$ to derive the expression

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big)\frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta}) + \sum_{j=1}^{P}\frac{\partial \mathbf{g}}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big)\frac{\partial \mathbf{v}^j}{\partial \mathbf{x}}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big)\frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta}) - \frac{\partial \dot{\mathbf{z}}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta}) \equiv \mathbf{0}_{N\times N}. \tag{41}$$

Now premultiply Eq. (41) by $e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger$, integrate the result over $[0, +\infty)$, and add it to Eq. (40). Then use Eqs. (26)–(28), the FNE analog of Eq. (20), assumption (B.5), and the fact that $\mathbf{z}(0;\boldsymbol{\beta}) \equiv \mathbf{x}_0$ implies $\partial \mathbf{z}(0;\boldsymbol{\beta})/\partial \mathbf{x}_0 \equiv \mathbf{I}_N$ to arrive at Eq. (31).

To prove Eq. (32), first differentiate Eq. (33) with respect to r using Leibniz's rule to get

$$\begin{aligned}\frac{\partial W^p}{\partial r}(\boldsymbol{\alpha}) = \int_0^{+\infty} e^{-rt}\Bigg[&-t f^p\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big)\\ &+ \frac{\partial f^p}{\partial \mathbf{x}}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big)\frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta})\\ &+ \sum_{j=1}^{P}\frac{\partial f^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big)\\ &\times\left[\frac{\partial \mathbf{v}^j}{\partial \mathbf{x}}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big)\frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta}) + \frac{\partial \mathbf{v}^j}{\partial r}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big)\right]\Bigg]dt.\end{aligned} \tag{42}$$

Then differentiate the identity form of Eq. (29) with respect to r to derive

$$\begin{aligned}&\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big)\frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta})\\ &+ \sum_{j=1}^{P}\frac{\partial \mathbf{g}}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big); \boldsymbol{\alpha}\big)\\ &\times\left[\frac{\partial \mathbf{v}^j}{\partial \mathbf{x}}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big)\frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta}) + \frac{\partial \mathbf{v}^j}{\partial r}\big(t, \mathbf{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}, r\big)\right]\\ &- \frac{\partial \dot{\mathbf{z}}}{\partial r}(t;\boldsymbol{\beta}) \equiv \mathbf{0}_{N\times 1}.\end{aligned} \tag{43}$$

Next, premultiply Eq. (43) by $e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger$, integrate the result over $[0,+\infty)$, and add it to Eq. (42). Then use Eqs. (26)–(28) and the FNE analog of Eq. (23) to simplify the resulting expression and arrive at

$$\frac{\partial W^p}{\partial r}(\boldsymbol{\beta}) = \int_0^{+\infty} e^{-rt}\Bigg[-tf^p\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}\big(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r\big);\boldsymbol{\alpha}\big)$$
$$+\sum_{j=1,j\neq p}^{P}\frac{\partial H^p}{\partial \mathbf{u}^j}\big(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}\big(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r\big),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha}\big)\frac{\partial \mathbf{v}^j}{\partial r}\big(t,\mathbf{z}(t;\boldsymbol{\beta});\boldsymbol{\alpha},r\big)\Bigg]dt$$
$$+\boldsymbol{\lambda}^p(0;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial r}(0;\boldsymbol{\beta}) - \lim_{t\to+\infty} e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta}). \tag{44}$$

Finally, employ assumption (B.5) and the fact that $\mathbf{z}(0;\boldsymbol{\beta})\equiv\mathbf{x}_0$ implies $\partial\mathbf{z}(0;\boldsymbol{\beta})/\partial r\equiv\mathbf{0}_N$ to reduce Eq. (44) to Eq. (32). □

Theorem 2 establishes the envelope theorem for FNE of discounted and autonomous infinite horizon differential games. Equations (30) and (32) characterize the effects of a perturbation in $\boldsymbol{\alpha}$ and r on the optimal value function of each player. They exhibit the same structure as their open-loop counterparts given in Eqs. (7) and (9), respectively. The first term on the right-hand side of Eqs. (30) and (32) is the direct or explicit effect that results from the explicit appearance of the parameters $\boldsymbol{\alpha}$ and r, while the second term on the right-hand side of Eqs. (30) and (32), is, as before, the strategic effect that results from the other $P-1$ players' response to the perturbation in the parameters $\boldsymbol{\alpha}$ and r. The main distinctive feature of the envelope theorem for a feedback information structure lies with Eq. (31). Contrary to the open-loop case, it demonstrates that the costate vector is equal to the shadow value of the initial state vector, the familiar envelope result from optimal control theory and a result fully consistent with the findings of [6].

In the remainder of the paper we apply Theorems 1 and 2 to a linear-quadratic class of differential games. The linear-quadratic structure permits the explicit derivation of an OLNE and a FNE and the corresponding envelope expressions. It also allows one to calculate and sign the strategic term in Eq. (8), thereby permitting a comparison of the dynamic pattern of the costate variable and the shadow value of the state variable for an OLNE.

5 A Linear-Quadratic Differential Game and Its OLNE

The linear-quadratic structure is one of the few for which an analytical solution for OLNE and FNE can be derived. Such a game is characterized by a linear state equation and a quadratic objective functional. This class of differential games is the work-horse of the field and has been widely applied in economics. For example, Miller and Salmon [15], Cohen and Michel [7], and Dockner and Neck [8] employ a linear-quadratic differential game to study various macroeconomic issues, Dockner and Long [9] and List and Mason [13] employ the structure to examine various matters in international pollution control, while Groot et al. [12] use it to study natural resource extracting firms.

We contemplate a linear-quadratic differential game with two players, one state variable, and one control variable for each player, as defined by

$$W^1(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{u^1(\cdot)} -\frac{1}{2}\int_0^{+\infty} e^{-rt}\Big[k_1\big[x(t)\big]^2 + k_2\big[u^1(t)\big]^2\Big]dt, \tag{45}$$

$$W^2(\boldsymbol{\beta}) \overset{\text{def}}{=} \max_{u^2(\cdot)} -\frac{1}{2}\int_0^{+\infty} e^{-rt}\left[l_1\left[x(t)\right]^2 + l_2\left[u^2(t)\right]^2\right]dt,$$
$$\text{s.t.} \quad \dot{x}(t) = ax(t) + bu^1(t) + cu^2(t), \qquad x(0) = x_0, \tag{46}$$

where $k_1, k_2, l_1, l_2, a, b, c$, and r are constant scalars, x_0 is the given initial value of the state variable, $\boldsymbol{\alpha} \overset{\text{def}}{=} (k_1, k_2, l_1, l_2, a, b, c)$, and $\boldsymbol{\beta} \overset{\text{def}}{=} (k_1, k_2, l_1, l_2, a, b, c, r, x_0)$. Apart from some minor differences in notation, the linear-quadratic class of differential games defined by Eqs. (45) and (46) is identical to that investigated by [10, Eqs. (7.1)–(7.3)]. Observe that we examine a subclass of linear-quadratic differential games in which only the square of the state variable and the square of the pth player's control variable appear in the integrand of the pth player's objective functional. Accordingly, the integrand function of the class is positively homogeneous of degree 2 in the state variable and given player's control variable. The said class can be generalized in various ways, as noted by [10, p. 171], but we prefer, as do they, to keep matters simple and not delve into such generalizations since they yield no additional insight to the matters at hand and only serve to complicate the analytical solution of the game. Following Dockner et al. [10, p. 173], we assume that the parameters k_1, k_2, l_1, l_2, and r are positive.

For the remainder of the paper, we do not impose specific economic content on the above game. Nonetheless, it is not difficult to provide an economic interpretation of the differential game with the proper signs of the parameters. For example, Dockner et al. [10, pp. 172–173] provide several different economic interpretations of the class of linear-quadratic differential games.

In this section and the next, we explicitly derive an OLNE and a FNE for the linear-quadratic differential game defined by Eqs. (45) and (46), and then calculate the envelope properties of the optimal value functions by partial differentiation. This procedure may be considered as a supportive example of Theorems 1 and 2. We begin by deriving the OLNE.

Let $(v^1(t;\boldsymbol{\beta}), v^2(t;\boldsymbol{\beta}))$ be an OLNE and $z(t;\boldsymbol{\beta})$ be its corresponding state variable path. The current-value Hamiltonians of the two players are then defined as

$$H^1\left(x, u^1, u^2, \lambda^1; \boldsymbol{\alpha}\right) \overset{\text{def}}{=} -\frac{1}{2}\left[k_1 x^2 + k_2\left[u^1\right]^2\right] + \lambda^1\left[ax + bu^1 + cu^2\right], \tag{47}$$

$$H^2\left(x, u^1, u^2, \lambda^2; \boldsymbol{\alpha}\right) \overset{\text{def}}{=} -\frac{1}{2}\left[l_1 x^2 + l_2\left[u^2\right]^2\right] + \lambda^2\left[ax + bu^1 + cu^2\right]. \tag{48}$$

By Eqs. (4) and (5), the necessary conditions are given by

$$u^1(t) = bk_2^{-1}\lambda^1(t), \tag{49}$$

$$u^2(t) = cl_2^{-1}\lambda^2(t), \tag{50}$$

$$\dot{\lambda}^1(t) = k_1 x(t) + [r-a]\lambda^1(t), \tag{51}$$

$$\dot{\lambda}^2(t) = l_1 x(t) + [r-a]\lambda^2(t). \tag{52}$$

In order to solve the necessary conditions, substitute Eqs. (49) and (50) into the state equation and combine it with the Eqs. (51) and (52). These actions result in the *canonical system*

$$\dot{x}(t) = ax(t) + b^2 k_2^{-1}\lambda^1(t) + c^2 l_2^{-1}\lambda^2(t), \tag{53}$$

$$\dot{\lambda}^1(t) = k_1 x(t) + [r-a]\lambda^1(t), \tag{54}$$

$$\dot{\lambda}^2(t) = l_1 x(t) + [r-a]\lambda^2(t). \tag{55}$$

The canonical system defined by Eqs. (53)–(55) may be written in matrix notation as $\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t)$, where $\mathbf{y}(t) \stackrel{\text{def}}{=} (x(t), \lambda^1(t), \lambda^2(t))^\dagger$ and

$$\mathbf{A} \stackrel{\text{def}}{=} \begin{bmatrix} a & b^2 k_2^{-1} & c^2 l_2^{-1} \\ k_1 & r-a & 0 \\ l_1 & 0 & r-a \end{bmatrix}. \tag{56}$$

In order to solve the canonical system, the eigenvalues μ_1, μ_2, and μ_3 of $\mathbf{A}$ are required. They are the solution to the characteristic equation $|\mathbf{A} - \mu \mathbf{I}_3| = [r-a-\mu]^2[a-\mu] - [r-a-\mu]M = 0$, where $\mathbf{I}_3$ is the identity matrix of order 3 and $M \stackrel{\text{def}}{=} c^2 l_1 l_2^{-1} + b^2 k_1 k_2^{-1} > 0$, seeing as k_1, k_2, l_1, and l_2 are positive. The eigenvalues are thus given by

$$\mu_1 = \frac{1}{2}\left[r - \sqrt{r^2 - 4a[r-a] + 4M}\right], \tag{57}$$

$$\mu_2 = \frac{1}{2}\left[r + \sqrt{r^2 - 4a[r-a] + 4M}\right], \tag{58}$$

$$\mu_3 = r - a. \tag{59}$$

Note that $r^2 - 4a[r-a] \equiv [r-2a]^2 \geq 0$, which, when combined with the fact that $M > 0$, implies that the eigenvalues are real.

We now employ two more assumptions before solving for the OLNE. First, assume that if the state equation is uncontrolled, i.e., if $u^1(t) \equiv 0$ and $u^2(t) \equiv 0$, then the resulting steady state solution, namely $x = 0$, is globally asymptotically stable, the latter of which is equivalent to $a < 0$. Second, assume that the steady-state solution of Eqs. (53)–(55), to wit, the origin, is simple, thereby implying that $|\mathbf{A}| \neq 0$, or equivalently, that all of the eigenvalues of $\mathbf{A}$ are nonzero. As a result, there are two possible cases for the sign pattern of the eigenvalues of $\mathbf{A}$ under the given assumptions, viz., (i) $\mu_1 > 0$, $\mu_2 > 0$, and $\mu_3 > 0$, or (ii) $\mu_1 < 0$, $\mu_2 > 0$, and $\mu_3 > 0$. Case (ii) is maintained in what follows, seeing as the steady state of the canonical system has a one-dimensional globally asymptotically stable manifold in this case, thereby implying that the steady state can be reached given an initial condition that lies on the said manifold. Because the product of the eigenvalues of a matrix equals its determinant, it follows that $|\mathbf{A}| < 0$ for case (ii).

Given the initial condition $x(0) = x_0$, the unique solution of Eqs. (53)–(55) can be found and then combined with Eqs. (49) and (50) to find the unique OLNE, which is found to be

$$v^1(t; \boldsymbol{\beta}) = x_0 b k_2^{-1} w_2 e^{\mu_1 t}, \qquad v^2(t; \boldsymbol{\beta}) = x_0 c l_2^{-1} w_3 e^{\mu_1 t}. \tag{60}$$

The corresponding state and costate time-paths are

$$z(t; \boldsymbol{\beta}) = x_0 e^{\mu_1 t}, \qquad \lambda^1(t; \boldsymbol{\beta}) = x_0 w_2 e^{\mu_1 t}, \qquad \lambda^2(t; \boldsymbol{\beta}) = x_0 w_3 e^{\mu_1 t}, \tag{61}$$

where $\mathbf{w} \stackrel{\text{def}}{=} (w_1, w_2, w_3)^\dagger \in \mathbb{R}^3$ is the eigenvector of $\mathbf{A}$ corresponding to the eigenvalue $\mu_1 < 0$, and which satisfies the following system of linear equations:

$$w_2 = -k_1[r-a-\mu_1]^{-1} < 0, \tag{62}$$

$$w_3 = -l_1[r - a - \mu_1]^{-1} < 0, \tag{63}$$

$$[a - \mu_1] + b^2 k_2^{-1} w_2 + c^2 l_2^{-1} w_3 = 0. \tag{64}$$

Note that without loss of generality we have set $w_1 \equiv 1$ as the normalization. Inspection of Eq. (61) shows that assumption (A.5) is satisfied. Given Eqs. (60) and (61), it is straightforward to calculate the optimal value function for each player, namely,

$$W^1(\boldsymbol{\beta}) = -\frac{1}{2} x_0^2 [r - 2\mu_1]^{-1} \left[k_1 + b^2 k_2^{-1} w_2^2\right] < 0, \tag{65}$$

$$W^2(\boldsymbol{\beta}) = -\frac{1}{2} x_0^2 [r - 2\mu_1]^{-1} \left[l_1 + c^2 l_2^{-1} w_3^2\right] < 0. \tag{66}$$

Armed with the preceding two expressions, one can explicitly calculate the envelope results given in Theorem 1. For player 1, say, the shadow value of the initial state is given by

$$\frac{\partial W^1(\boldsymbol{\beta})}{\partial x_0} = -x_0 [r - 2\mu_1]^{-1} \left[k_1 + b^2 k_2^{-1} w_2^2\right] < 0, \tag{67}$$

with an analogous result for player 2. By Eqs. (61) and (62), the corresponding initial value of the costate variable is $\lambda^1(0; \boldsymbol{\beta}) = x_0 w_2 < 0$. Note that $\lambda^1(0; \boldsymbol{\beta})$ and $\partial W^1(\boldsymbol{\beta})/\partial x_0$ have the same sign but are not equal—the latter is shown below and is as predicted by Theorem 1.

The linear-quadratic structure provides the opportunity to explicitly show the envelope results in Theorem 1. For player 1, for example, the right-hand side of Eq. (8) can be explicitly calculated using the Hamiltonian of player 1, i.e., Eq. (47), the OLNE given in Eq. (60), and its corresponding state and costate paths displayed in Eq. (61), and is found to be

$$\begin{aligned} &\lambda^1(0; \boldsymbol{\beta}) + \int_0^{+\infty} e^{-rt} \left[\frac{\partial H^1}{\partial u^2} \left(z(t; \boldsymbol{\beta}), v^1(t; \boldsymbol{\beta}), v^2(t; \boldsymbol{\beta}), \lambda^1(t; \boldsymbol{\beta}); \boldsymbol{\alpha}\right) \frac{\partial v^2}{\partial x_0}(t; \boldsymbol{\beta}) \right] dt \\ &\quad = x_0 w_2 + x_0 [r - 2\mu_1]^{-1} c^2 l_2^{-1} w_2 w_3. \end{aligned} \tag{68}$$

Note that the strategic effect is positive, i.e., $x_0[r - 2\mu_1]^{-1} c^2 l_2^{-1} w_2 w_3 > 0$, as $\mu_1 < 0$, $w_2 < 0$, and $w_3 < 0$, while all the other parameters in it are positive. Substituting Eqs. (62) and (64) in Eq. (68), it follows that

$$\begin{aligned} &\lambda^1(0; \boldsymbol{\beta}) + \int_0^{+\infty} e^{-rt} \left[\frac{\partial H^1}{\partial u^2} \left(z(t; \boldsymbol{\beta}), v^1(t; \boldsymbol{\beta}), v^2(t; \boldsymbol{\beta}), \lambda^1(t; \boldsymbol{\beta}); \boldsymbol{\alpha}\right) \frac{\partial v^2}{\partial x_0}(t; \boldsymbol{\beta}) \right] dt \\ &\quad = x_0 w_2 [r - 2\mu_1]^{-1} \left[[r - 2\mu_1] + c^2 l_2^{-1} w_3\right] \\ &\quad = -x_0 [r - 2\mu_1]^{-1} k_1 [r - a - \mu_1]^{-1} \left[[r - a - \mu_1] + a - \mu_1 + c^2 l_2^{-1} w_3\right] \\ &\quad = -x_0 [r - 2\mu_1]^{-1} k_1 [r - a - \mu_1]^{-1} \left[[r - a - \mu_1] - b^2 k_2^{-1} w_2\right] \\ &\quad = -x_0 [r - 2\mu_1]^{-1} \left[k_1 + b^2 k_2^{-1} w_2^2\right] = \frac{\partial W^1(\boldsymbol{\beta})}{\partial x_0} < 0, \end{aligned} \tag{69}$$

with the last equality following from Eq. (67). An analogous result can be derived for player 2. Equation (69) confirms Eq. (8) of Theorem 1. The derivations of the remaining envelope results in Theorem 1 are similar and so are omitted.

The linear-quadratic structure also yields two results that cannot be obtained in the general case. The first follows from Eqs. (68) and (69), scilicet, that the costate variable $\lambda^1(0; \boldsymbol{\beta})$

is less than, i.e., is more negative than, the shadow value of the initial state $\partial W^1(\boldsymbol{\beta})/\partial x_0$, seeing as the strategic effect is positive under the given assumptions. In other words, Eqs. (68) and (69) show that $\lambda^1(0;\boldsymbol{\beta}) < \partial W^1(\boldsymbol{\beta})/\partial x_0 < 0$. Hence, the initial value of the costate variable overstates the decrease in the optimal value function of player 1 due to an increase in the initial state. Thus the strategic effect is beneficial to player 1, in that it partially offsets the decline in the optimal value function of player 1 due to the increase in the initial state.

The second result generated by the linear-quadratic structure is that the growth rates of the costate variable and shadow value of the state variable are the same. To see this, consider a truncated but otherwise identical version of the infinite horizon differential game defined in Eqs. (45) and (46), in which the fixed but arbitrary initial time is $s \in [0, +\infty)$ and the discount factor is $\exp(-r[t-s])$. Given that the OLNE is played from time 0 until time s, the initial value of the state variable in the truncated game is given by $x(s) = x_s \stackrel{\text{def}}{=} x_0 e^{\mu_1 s}$ from Eq. (61). Then by Theorem 4.3 of [10], the OLNE of the original game is also the OLNE of the truncated game in view of the fact that the OLNE is time consistent. Accordingly, by either Eq. (67) or Eq. (69), the shadow value of the initial state at time s in the truncated game for player 1 is given by

$$\frac{\partial W^1}{\partial x_s}(\boldsymbol{\alpha}, r, x_s)\bigg|_{x_s = x_0 e^{\mu_1 s}} = -x_s[r-2\mu_1]^{-1}\left[k_1 + b^2 k_2^{-1} w_2^2\right]\Big|_{x_s = x_0 e^{\mu_1 s}}$$
$$= -x_0 e^{\mu_1 s}[r-2\mu_1]^{-1}\left[k_1 + b^2 k_2^{-1} w_2^2\right]. \tag{70}$$

By Eq. (61), the value of the costate variable at time s for player 1 in the truncated game is

$$\lambda^1(t-s;\boldsymbol{\alpha}, r, x_s)\big|_{\substack{x_s = x_0 e^{\mu_1 s} \\ t=s}} = x_s w_2 e^{\mu_1[t-s]}\big|_{\substack{x_s = x_0 e^{\mu_1 s} \\ t=s}} = x_0 w_2 e^{\mu_1 s}. \tag{71}$$

Examination of Eqs. (70) and (71) shows that the growth rate of the shadow value of the initial state at time s is the same as the growth rate of the costate variable at time s. Given that $s \in [0, +\infty)$ is fixed but otherwise arbitrary, this proves that the aforesaid growths rate are identical over the entire planning horizon. Again, a fully analogous result can be obtained for player 2.

The preceding two paragraphs demonstrate that the main problem with treating the initial value of the costate variable as the shadow value of the initial state is the magnitude of the difference in their values for the class of linear-quadratic games under consideration. In order to get some sense for the magnitudes involved in comparing the costate variable and the shadow value of the state, assume the following values for the parameters: $k_1 = l_1 = 1$, $k_2 = l_2 = 0.75$, $a = -0.25$, $b = c = 1$, $r = 0.05$, and $x_0 = 25$. Figure 1 depicts the time-paths of the costate variable and the shadow value for player 1. The numerical results show that the costate variable is approximately 25 % lower than the shadow value of the state at the initial time, but that it converges to the shadow value in the limit of the planning horizon as the steady state is approached.

The above results have a policy implication. In oligopolistic resource extracting models of the firm, for example, if a policy maker wants to protect the resource stock by regulating the extraction rates of the firms, then the shadow value of the stock must be determined. Unfortunately, given an open-loop information structure, the regulator will overestimate the shadow value of the resource stock if the costate variable is used instead of the shadow value of the stock. As a result, the policy maker will "over-regulate" the extraction rates and hence reduce the welfare of the economy. In passing, it is important to remember that the above results do not hold in general, but are limited to the linear-quadratic class of games under consideration.

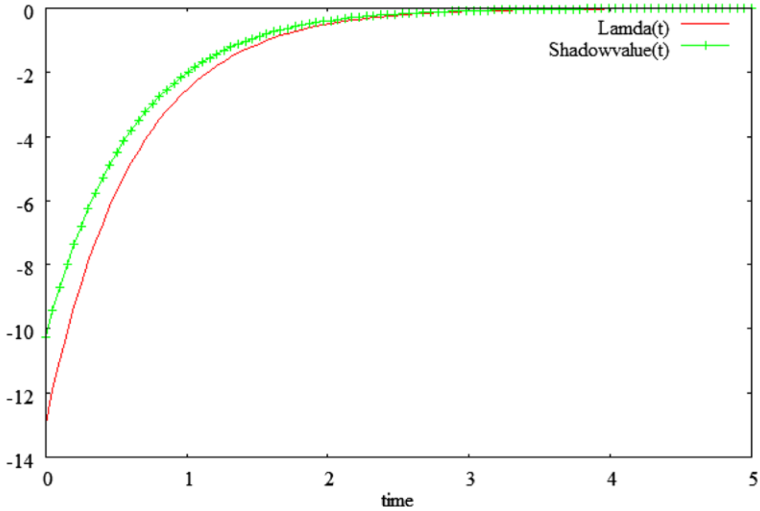

Fig. 1 The costate variable and the shadow value of the state variable in a linear-quadratic differential game

6 A Linear-Quadratic Differential Game and Its FNE

Now we turn to the derivation of a FNE of the game defined in Eqs. (45) and (46). The Hamilton-Jacobi-Bellman (HJB) equations for this game are

$$rW^1(x;\boldsymbol{\alpha},r) = \max_{u^1}\left\{-\frac{1}{2}\left[k_1x^2 + k_2\left[u^1\right]^2\right] + \frac{\partial W^1}{\partial x}(x;\boldsymbol{\alpha},r)\left[ax + bu^1 + cu^2\right]\right\}, \tag{72}$$

$$rW^2(x;\boldsymbol{\alpha},r) = \max_{u^2}\left\{-\frac{1}{2}\left[l_1x^2 + l_2\left[u^2\right]^2\right] + \frac{\partial W^2}{\partial x}(x;\boldsymbol{\alpha},r)\left[ax + bu^1 + cu^2\right]\right\}, \tag{73}$$

where x is any given admissible value of the state variable. In order to derive a FNE, one must first solve Eqs. (72) and (73) to find the optimal value function of each player. To this end, we propose a general functional form for $W^p(\cdot)$ containing an unknown parameter, and then seek to determine the values of the two parameters such that the proposed functional form for $W^p(\cdot)$ satisfies Eqs. (72) and (73). This information is then used to derive a FNE.

Given that the game under consideration is linear-quadratic, the method of undetermined coefficients leads to the hypothesis that the optimal value functions are of the form

$$W^p(x;\boldsymbol{\alpha},r) = \frac{1}{2}\phi^p x^2, \quad p = 1, 2, \tag{74}$$

where ϕ^p, $p = 1, 2$, are the unknown parameters of the optimal value functions to be determined. Substituting Eq. (74) into the HJB equations, the first-order necessary conditions associated with Eqs. (72) and (73) are given by

$$u^1 = bk_2^{-1}\phi^1 x, \tag{75}$$

$$u^2 = cl_2^{-1}\phi^2 x. \tag{76}$$

Substituting Eqs. (75) and (76) back into the HJB equations, we get a standard system of algebraic Riccati equations, namely,

$$k_1 + [r - 2a]\phi^1 - b^2 k_2^{-1}\big[\phi^1\big]^2 - 2c^2 l_2^{-1}\phi^1\phi^2 = 0, \tag{77}$$

$$l_1 + [r - 2a]\phi^2 - c^2 l_2^{-1}\big[\phi^2\big]^2 - 2b^2 k_2^{-1}\phi^1\phi^2 = 0. \tag{78}$$

Given that the parameters k_1, k_2, l_1, and l_2 are positive and that $a < 0$, Theorem 1 of [11] implies that Eqs. (77) and (78) admit a unique negative solution, that is, $\phi^1 < 0$ and $\phi^2 < 0$, both of which depend on the parameters of the game.

Substituting Eqs. (75) and (76) in the state equation yields

$$\dot{x}(t) = \big[a + b^2 k_2^{-1}\phi^1 + c^2 l_2^{-1}\phi^2\big]x(t), \qquad x(0) = x_0. \tag{79}$$

By Theorem 2.4 of [2], there exists a unique and absolutely continuous solution of the initial value problem given in Eq. (79), which is denoted by $z(t; \boldsymbol{\beta}) = x_0 e^{\eta t}$, where $\eta \stackrel{\text{def}}{=} a + b^2 k_2^{-1}\phi^1 + c^2 l_2^{-1}\phi^2 < 0$. Furthermore, $z(t; \boldsymbol{\beta}) = x_0 e^{\eta t}$ lives in the compact interval $[0, x_0]$, seeing as $\eta < 0$. Hence the quadratic optimal value functions $W^p(\cdot)$ are bounded. Therefore, using Eqs. (75) and (76), the strategies

$$v^1(x; \boldsymbol{\alpha}, r) = b k_2^{-1}\phi^1 x, \qquad v^2(x; \boldsymbol{\alpha}, r) = c l_2^{-1}\phi^2 x \tag{80}$$

constitute a (subgame perfect) FNE by Theorem 4.4 of [10].

By using $z(t; \boldsymbol{\beta}) = x_0 e^{\eta t}$, it follows that the open-loop representation of the control paths generated by the FNE is given by

$$\tilde{v}^1(t; \boldsymbol{\beta}) = x_0 b k_2^{-1}\phi^1 e^{\eta t}, \qquad \tilde{v}^2(t; \boldsymbol{\beta}) = x_0 c l_2^{-1}\phi^2 e^{\eta t}, \tag{81}$$

and the resulting optimal value functions of the two players are

$$W^1(\boldsymbol{\beta}) = -\frac{1}{2}x_0^2[r - 2\eta]^{-1}\big[k_1 + b^2 k_2^{-1}\big[\phi^1\big]^2\big] < 0, \tag{82}$$

$$W^2(\boldsymbol{\beta}) = -\frac{1}{2}x_0^2[r - 2\eta]^{-1}\big[l_1 + c^2 l_2^{-1}\big[\phi^2\big]^2\big] < 0. \tag{83}$$

Accordingly, the shadow value of the initial state x_0 for player 1 is

$$\frac{\partial W^1(\boldsymbol{\beta})}{\partial x_0} = -x_0[r - 2\eta]^{-1}\big[k_1 + b^2 k_2^{-1}\big[\phi^1\big]^2\big] < 0, \tag{84}$$

with a similar result holding for player 2.

Next, we explicitly derive one of the envelope results for the FNE, say, Eq. (31) in Theorem 2. In contrast to the open-loop case, the costate variable $\lambda^p(t)$ is not explicitly specified in the HJB equations that formed the basis for the preceding derivation of the FNE. In order to get an explicit expression for the costate variable corresponding to the FNE, we turn to the open-loop representations of the current-value Hamiltonians generated by the FNE.

Accordingly, the current-value Hamiltonian of, say, player 1, is given by

$$H^1\big(x, u^1, v^2(x; \boldsymbol{\alpha}, r), \lambda^1; \boldsymbol{\alpha}\big) \stackrel{\text{def}}{=} -\frac{1}{2}\big[k_1 x^2 + k_2\big[u^1\big]^2\big] + \lambda^1\big[ax + bu^1 + cv^2(x; \boldsymbol{\alpha}, r)\big]. \tag{85}$$

In this case the necessary condition in Eq. (27) becomes

$$\frac{\partial H^1}{\partial u^1}\big(x, u^1, v^2(x;\boldsymbol{\alpha}, r), \lambda^1;\boldsymbol{\alpha}\big) = 0 \quad \Rightarrow \quad \lambda^1 = k_2 b^{-1} u^1. \tag{86}$$

By substituting the open-loop representation of the control path generated by the FNE given in Eq. (81), the explicit time-path of the costate variable corresponding to the FNE is found to be

$$\lambda^1(t;\boldsymbol{\beta}) = x_0 \phi^1 e^{\eta t}, \tag{87}$$

hence $\lambda^1(0;\boldsymbol{\beta}) = x_0\phi^1$.

In order to show that $\partial W^1(\boldsymbol{\beta})/\partial x_0 = \lambda^1(0;\boldsymbol{\beta})$, first rearrange Eq. (77) to get

$$\begin{aligned} k_1 + b^2 k_2^{-1}\big[\phi^1\big]^2 &= -r\phi^1 + 2a\phi^1 + 2b^2 k_2^{-1}\big[\phi^1\big]^2 + 2c^2 l_2^{-1}\phi^1\phi^2 \\ &= -r\phi^1 + 2\phi^1\big[a + b^2 k_2^{-1}\phi^1 + c^2 l_2^{-1}\phi^2\big]. \end{aligned} \tag{88}$$

Then multiply Eq. (88) by x_0 and use the definition $\eta \overset{\text{def}}{=} a + b^2 k_2^{-1}\phi^1 + c^2 l_2^{-1}\phi^2 < 0$ to simplify the resulting expression and arrive at

$$-x_0[r - 2\eta]^{-1}\big[k_1 + b^2 k_2^{-1}\big[\phi^1\big]^2\big] = x_0\phi^1. \tag{89}$$

Inspection of Eqs. (84) and (89) and recalling that $\lambda^1(0;\boldsymbol{\beta}) = x_0\phi^1$ confirms Eq. (31) in Theorem 2. That is, we have explicitly shown that $\lambda^1(0;\boldsymbol{\beta})$, the initial value of the costate variable, is identically equal to the shadow value of the initial state variable, namely, $\partial W^1(\boldsymbol{\beta})/\partial x_0$, for the FNE of the linear-quadratic differential game defined by Eqs. (45) and (46), thereby permitting the conventional economic interpretation of the costate variable in this case.

7 Summary and Conclusions

We have successfully extended the envelope theorem so as to apply to the class of autonomous and discounted infinite horizon differential games. The presence of strategic effects in all but one of the envelope expressions in Theorems 1 and 2 requires that the equilibrium of the game must be known in order to sign the said expressions. This fact limits the application of the envelope results in differential games relative to optimal control problems, as explicit equilibria are derivable but for a limited class of differential games. Even so, a comparison of Theorems 1 and 2 shows that when the information structure is of the feedback variety, the initial value of the costate vector equals the shadow value of the initial state vector, thereby permitting the conventional interpretation of the costate vector for FNE, but not for OLNE.

Arguably, the most important contribution of the paper is the application of the general envelope theorems to the work-horse class of differential games, videlicet, the linear-quadratic class. By computing an OLNE and a FNE of the aforesaid game, we were able to derive explicit expressions for the general envelope results given in Theorems 1 and 2. In addition, the linear-quadratic structure permitted the derivation of two results that cannot be obtained in the general setting. Specifically, it was shown that, given an open-loop information structure, the costate variable uniformly overstates the shadow value of the state variable, but that the growth rates of the two are identical.

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