The Intrinsic Comparative Dynamics of Locally Differentiable Feedback Nash Equilibria of Autonomous and Exponentially Discounted Infinite Horizon Differential Games

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Abstract

The comparative dynamics of locally differentiable feedback Nash equilibria are derived for the ubiquitous class of autonomous and exponentially discounted infinite horizon differential games. The resulting refutable implications are intrinsic to the said class of differential games, and thus form their basic, empirically testable, properties. Their relationship with extant results in optimal control theory and static game theory is discussed. Separability conditions are identified on the instantaneous payoff and transition functions under which the intrinsic comparative dynamics collapse, in form, to those in optimal control problems. Applications of the results to capital accumulation and sticky-price games are provided.

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1. Introduction

The derivation of the basic, fundamental, or intrinsic comparative statics of an economic model—the archetypal utility maximization problem—can be traced back to Antonelli (1886) and Slutsky (1915). But it was Samuelson (1947, p. 32) who first derived the intrinsic comparative statics of the class of differentiable, unconstrained optimization problems. Twenty-seven years later Silberberg (1974) provided an extension of Samuelson's (1947) theorem by deriving the basic comparative statics of the class of differentiable, constrained optimization problems. More than three decades then passed before Partovi and Caputo (2006, 2007) further generalized and unified the approach to deriving the fundamental comparative statics of the class of differentiable, constrained optimization problems using the concept of generalized compensated derivatives.

It is important to remark at this juncture that the intrinsic comparative statics of differentiable optimization problems are defined as those qualitative comparative statics properties that follow solely from the assumption that a locally differentiable solution exists. The lack of any other assumptions on the optimization problem is what makes such properties basic, fundamental, or intrinsic. Although it is nearly universal, and indeed often sensible and justifiable, to make assumptions regarding the functional forms of the objective or constraint functions, or suppose that certain monotonicity or curvature properties hold on the said functions, these and other such ad hoc suppositions transcend those required to derive the fundamental comparative statics of a differentiable optimization problem. Accordingly, such assumptions do not yield intrinsic qualitative results and thus are not made in this work.

A similar development to that in static optimization took place in continuous time dynamic optimization, although it was initiated 43 years after the work of Samuelson (1947). In particular, Caputo (1990a) derived the basic comparative dynamics of an open-loop solution for a general class of variational calculus problems, while Caputo (1990b) did the same for an open-loop solution of a general class of optimal control problems. More than a decade later, Caputo (2003) essentially "closed the loop" by deriving the fundamental comparative dynamics of a closed-loop (or feedback) solution of the class of autonomous and exponentially discounted infinite horizon optimal control problems.

Another similar development took place in static (or atemporal) game theory, but this one was initiated 49 years after Samuelson's (1947) basic contribution. In this case, Caputo (1996) derived the intrinsic comparative statics of a general class of static games possessing locally differentiable Nash equilibria. It took only two more years before Caputo (1998) did the same for atemporal Stackelberg equilibria.

Given the preceding historical evolution, the goal of this paper is a wholly natural one, to wit, the derivation of the intrinsic comparative dynamics of locally differentiable feedback Nash equilibria for a ubiquitous class of differential games. The class of differential games contemplated are autonomous, exponentially discounted, have an infinite time horizon, and possess a feedback information structure for which its corresponding feedback Nash equilibria exist. This class of differential games is, arguably, the most widely employed and studied in economics. Inasmuch as the goal of the paper is a natural extension of the aforementioned literature, little motivation need be provided. Nonetheless, several remarks are worthwhile to make at this stage.

First, given that it is more difficult to derive a closed-form solution for differential games than for optimal control problems or static games, the results obtained herein go some way towards overcoming the fact that "It is difficult to obtain results from general differential games," [Reinganum (1982, p. 674)], seeing as the comparative dynamics results are obtained in a rather general setting. Second, the main result shows that all differential games of the said class possess refutable, and thus in principle, empirically testable comparative dynamics properties. Third, these properties are intrinsic to the studied class of differential games and thus should be tested or imposed in empirical work that uses differential game theory as the basis for the development of an empirical model. And fourth, the intrinsic comparative dynamics take the form, in part, of generalized Slutsky-like expressions, i.e., one portion of the basic comparative dynamics take the form of linear combinations of partial derivatives of the feedback Nash equilibria with respect to the parameters and state variables of the game, while the other portion reflects the impact of the strategic nature of the differential game on its basic comparative dynamics.

As is often the case when one studies a general class of problems—and there is no exception in the present instance—certain structural elements of the class point to special cases which are wholly unanticipated, yet yield insight not possible in a typical tightly specified setting, and which have real utility in applied work. In particular, it is shown that if the instantaneous payoff function of every player is additively separable between that player's control variables and those of every other player, and if, in addition, all the transition functions are additively separable between the control variables of the different players, then the intrinsic comparative dynamics of differential games are of the same form as those in optimal control problems. Thus, even though a differential game is inherently strategic in nature, the above restrictions lead to intrinsic comparative dynamics in which the strategic element is absent. Moreover, the separability restrictions are common to many of the applied differential games appearing in the literature, as is documented in §3. Indeed, even the seemingly ubiquitous and workhorse class of linear-quadratic differential games satisfies the aforesaid separability conditions, thereby implying that many of the applied differential games solved in the literature possess intrinsic comparative dynamics in which the strategic component vanishes. Applications of the results to generalized capital accumulation and sticky-price games round out the contribution of the manuscript, and at the same time, impart some economic intuition to the results.

Finally, note that the following notational conventions are adopted: (i) all vectors are column vectors and are indicated in boldface type, (ii) the derivative of a scalar-valued function with respect to a column vector is a row vector, and is indicated by a boldface subscript letter on the function, (iii) the derivative of a vector-valued function with respect to a vector is a Jacobian matrix, the number of rows of which equal the number of functions being differentiated and the number of columns of which equal the number of elements in the vector that the derivative is taken with respect to, and is indicated by a boldface subscript letter on the boldface function, and (iv) the symbol "†" indicates transposition.

2. Technical Preliminaries

Consider the class of exponentially discounted and autonomous differential games consisting of a finite number $P \in \mathbb{Z}_{++}$ of players, indexed by $p \in \{1,2,\ldots,P\}$, and played over a infinite time horizon, where $t \in [0,+\infty)$ is a fixed but arbitrary initial time, often referred to as a base time. The state of the differential game at each instant $\tau \in [t,+\infty)$ is given by the state vector $\mathbf{x}(\tau) \in \mathrm{X}$, where $\mathrm{X} \subseteq \mathbb{R}^N$ is an open set referred to as the state space of the game. The initial value of the state vector, denoted by $\mathbf{x}(t)$, is fixed at the value $\mathbf{x}_t \in \mathrm{X}$, and is thus parametric to the players. In contrast, no conditions are placed on the limiting value of the state vector, i.e., $\lim_{\tau\to+\infty} \mathbf{x}(\tau)$ is free or unrestricted. Superscripts placed on functions, variables, or parameters are used to denote player specific entities. At every instant $\tau \in [t,+\infty)$ of the game, each player $p \in \{1,2,\ldots,P\}$ chooses the value $\mathbf{u}^p(\tau) \in \mathbb{R}^{M^p}$ of the vector-valued control function $\mathbf{u}^p(\cdot)$ to maximize their payoff functional. The state of the differential game evolves according to the system of autonomous differential equations $\dot{\mathbf{x}}(\tau) = \mathbf{g}\left(\mathbf{x}(\tau),\mathbf{u}^1(\tau),\mathbf{u}^2(\tau),\ldots,\mathbf{u}^P(\tau);\boldsymbol{\alpha}\right)$, where $\boldsymbol{\alpha} \in \mathbb{R}^K$ is a vector of time-independent parameters that enters the transition function $\mathbf{g}(\cdot)$ and the instantaneous payoff function $f^p(\cdot)$ of each player $p \in \{1,2,\ldots,P\}$. Define $r^p \in \mathbb{R}_{++}$ as the rate of discount used by player $p \in \{1,2,\ldots,P\}$ in exponentially discounting the instantaneous payoff function $f^p(\cdot)$, and also define $\mathbf{r}^\dagger \stackrel{\text{def}}{=} (r^1,r^2,\ldots,r^P) \in \mathbb{R}^P_{++}$ as the vector of discount rates in the differential game. Furthermore, no constraints are placed on the state variables or the control variables of any player. Finally, and for the purpose of notational clarity, define $\mathbf{u}^{-p}(\tau) \stackrel{\text{def}}{=} \left(\mathbf{u}^1(\tau)^\dagger,\mathbf{u}^2(\tau)^\dagger,\ldots,\mathbf{u}^{p-1}(\tau)^\dagger,\mathbf{u}^{p+1}(\tau)^\dagger,\ldots,\mathbf{u}^P(\tau)^\dagger\right)^\dagger$ as the column vector consisting of the value of all the players' control functions at time $\tau \in [t,+\infty)$, save for player p.

Putting the above information together, the class of differential games under consideration is given by

$$V^p(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{\mathbf{u}^p(\cdot)} \int_t^{+\infty} e^{-r^p[\tau-t]} f^p\left(\mathbf{x}(\tau),\mathbf{u}^1(\tau),\mathbf{u}^2(\tau),\ldots,\mathbf{u}^P(\tau);\boldsymbol{\alpha}\right)d\tau \tag{1}$$

$$\text{s.t. } \dot{\mathbf{x}}(\tau) = \mathbf{g}\left(\mathbf{x}(\tau),\mathbf{u}^1(\tau),\mathbf{u}^2(\tau),\ldots,\mathbf{u}^P(\tau);\boldsymbol{\alpha}\right),\ \mathbf{x}(t) = \mathbf{x}_t,$$

for $p \in \{1,2,\ldots,P\}$, where $\boldsymbol{\beta} \overset{\text{def}}{=} (\mathbf{x}_t^\dagger, \boldsymbol{\alpha}^\dagger, \mathbf{r}^\dagger)^\dagger \in X \times \mathbb{R}^K \times \mathbb{R}_{++}^P$ is the vector of parameters of the game, and $V^p(\cdot)$ is the current value optimal value function—current value function hereafter—of player p. Seeing as τ only explicitly enters differential game (1) through the exponential discount function and the time horizon is infinite, $V^p(\cdot)$ is not an explicit function of the fixed but arbitrary base time $t \in [0,+\infty)$. The following assumptions are placed on the differential game defined in Eq. (1):

(A.1) The payoff functions $f^p(\cdot): \mathbb{R}^N \bigtimes_{p=1}^{P} \mathbb{R}^{M^p} \times \mathbb{R}^K \to \mathbb{R}$, $p \in \{1,2,\ldots,P\}$, and the transition functions $g^n(\cdot): \mathbb{R}^N \bigtimes_{p=1}^{P} \mathbb{R}^{M^p} \times \mathbb{R}^K \to \mathbb{R}$, $n = 1,2,\ldots,N$, are $C^{(2)}$ on their domains. Furthermore, the payoff functions $f^p(\cdot)$, $p \in \{1,2,\ldots,P\}$, are bounded.

(A.2) The information structure of the game is a feedback pattern and therefore consists of the set $\{\mathbf{x}(\tau), \boldsymbol{\alpha}, \mathbf{r}\}$ at time $\tau \in [t,+\infty)$ for every player $p \in \{1,2,\ldots,P\}$.

(A.3) There exists a feedback Nash equilibrium of differential game (1) for each $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$, designated by the P-tuple of vectors

$$\mathbf{v}(\mathbf{z}(\tau;\boldsymbol{\beta});\boldsymbol{\alpha},\mathbf{r}) \overset{\text{def}}{=} \left(\mathbf{v}^1(\mathbf{z}(\tau;\boldsymbol{\beta});\boldsymbol{\alpha},\mathbf{r})^\dagger, \mathbf{v}^2(\mathbf{z}(\tau;\boldsymbol{\beta});\boldsymbol{\alpha},\mathbf{r})^\dagger, \ldots, \mathbf{v}^P(\mathbf{z}(\tau;\boldsymbol{\beta});\boldsymbol{\alpha},\mathbf{r})^\dagger\right)^\dagger \in \bigtimes_{p=1}^{P} \mathbb{R}^{M^p},$$

where $\mathbf{z}(\tau;\boldsymbol{\beta}) \in X \subseteq \mathbb{R}^N$ is the associated trajectory of the state vector, $\boldsymbol{\lambda}^p(\tau;\boldsymbol{\beta}) \in \mathbb{R}^N$, $p \in \{1,2,\ldots,P\}$, is the corresponding trajectory of the current value costate vector of player p, $\boldsymbol{\lambda}(\tau;\boldsymbol{\beta}) \overset{\text{def}}{=} \left(\boldsymbol{\lambda}^1(\tau;\boldsymbol{\beta})^\dagger, \boldsymbol{\lambda}^2(\tau;\boldsymbol{\beta})^\dagger, \ldots, \boldsymbol{\lambda}^P(\tau;\boldsymbol{\beta})^\dagger\right)^\dagger \in \mathbb{R}^{NP}$, and $B(\boldsymbol{\beta}^\circ;\delta)$ is an open $(N+A+P)$-ball centered at the given value $\boldsymbol{\beta}^\circ \overset{\text{def}}{=} (\mathbf{x}_t^{\circ\dagger}, \boldsymbol{\alpha}^{\circ\dagger}, \mathbf{r}^{\circ\dagger})^\dagger \in X \times \mathbb{R}^K \times \mathbb{R}_{++}^P$ of radius $\delta > 0$.

(A.4) A feedback Nash equilibrium function $\mathbf{v}(\cdot) \in C^{(1)}$ on the open set $\Omega \subset X \times \mathbb{R}^K \times \mathbb{R}_{++}^P$, where $\Omega \overset{\text{def}}{=} \left\{(\mathbf{x},\boldsymbol{\alpha},\mathbf{r}) : (\mathbf{x},\boldsymbol{\alpha},\mathbf{r}) = (\mathbf{z}(\tau;\boldsymbol{\beta}),\boldsymbol{\alpha},\mathbf{r}) \forall \tau \in [t,+\infty) \wedge \boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)\right\}$.

(A.5) The current value function $V^p(\cdot) \in C^{(2)}$, $p \in \{1,2,\ldots,P\}$, on the open set Ω.

The smoothness supposition in assumption (A.1) is required in view of the fact that a differential characterization of the comparative dynamics of feedback Nash equilibria is sought.

The bounded nature of every instantaneous payoff function, in conjunction with the exponential discount function, implies that the objective functional in the optimal control problem that defines a feedback Nash strategy of player p —given in Eq. (2) below—converges for all playable pairs of functions $\left(\mathbf{x}(\cdot),\mathbf{u}^{p}(\cdot)\right)$ for all $p \in \{1,2,\ldots,P\}$. Thus the maximization problem defining each player's feedback Nash strategy is well defined given the infinite time horizon of the game.

Assumption (A.2) makes explicit the information that is available to the players at each point in time of the game. In particular, it is assumed that at time $\tau \in [t,+\infty)$, every player knows the value of the state vector at time τ, the parameter vector $\boldsymbol{\alpha}$, and the discount rate vector $\mathbf{r}$, an information pattern that is a slight generalization of that defined as feedback by Basar and Olsder [1999, Def. 5.6(v)], the generalization being the addition of the parameter vector $\boldsymbol{\alpha}$ and the discount rate vector $\mathbf{r}$ to the information structure. The feedback information structure and the autonomous nature of the differential game defined by Eq. (1) implies that the *p*th player's optimal control at time τ is an *explicit* function of the value of the state vector at time τ, i.e., $\mathbf{x}(\tau)$, the parameter vector $\boldsymbol{\alpha}$, and the discount rate vector $\mathbf{r}$, but not the base time t or the initial value of the state vector $\mathbf{x}_t$. The dependence of the *p*th player's optimal control at time τ on the parameters $(t,\mathbf{x}_t)$ therefore occurs indirectly, via the equilibrium trajectory of the state vector $\mathbf{z}(\tau;\boldsymbol{\beta})$.

Assumption (A.3) asserts the existence of a feedback Nash equilibrium for all values of the parameters in some open ball, and is a result of the facts that the information structure of the game is of the feedback variety and the class of differential games defined by Eq. (1) is generic. Alternatively, one could impose assumptions on the primitives that imply assumption (A.3). This approach, however, has a nontrivial disadvantage, to wit, it implies that the resulting comparative dynamics are not intrinsic to the differential game but are instead conditioned on sufficient conditions that transcend those implied by directly assuming the existence of a feedback Nash equilibrium. Consequently, such assumptions are not made, as they reduce the generality and applicability of the results.

Assumptions (A.4) and (A.5) are crucial to the foregoing analysis and are adopted because a differential characterization of the intrinsic comparative dynamics of feedback Nash equilibria of the differential game defined in Eq. (1) is sought. The stated differentiability is local in nature with respect to the parameter vector $(\boldsymbol{\alpha},\mathbf{r})$ and with respect to the value of the state vector $\mathbf{x}$. If, however, $\mathbf{v}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r})$ is a feedback Nash equilibrium for all $\mathbf{x}\in X$ and for all $(\boldsymbol{\alpha},\mathbf{r})\in B\big((\boldsymbol{\alpha}^{\circ},\mathbf{r}^{\circ});\varepsilon\big)$, then by Theorem 4.4 of Dockner et al. (2000), $\mathbf{v}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r})$ is also subgame perfect, i.e., it is a subgame perfect feedback Nash equilibrium, also referred to as a Markov perfect Nash equilibrium in the literature. If this alternative assumption holds, then it is natural to extend suppositions (A.4) and (A.5) by instead stipulating that $\mathbf{v}(\cdot)\in C^{(1)}$ and $V^{p}(\cdot)\in C^{(2)}$ for $p\in\{1,2,\ldots,P\}$, for all $\mathbf{x}\in X$ and for all $(\boldsymbol{\alpha},\mathbf{r})\in B\big((\boldsymbol{\alpha}^{\circ},\mathbf{r}^{\circ});\varepsilon\big)$, so that the differentiability extends over the entire state space—as opposed to just an open neighborhood of the equilibrium path of the state vector $\mathbf{z}(\tau;\boldsymbol{\beta})\in X$ —in order to conform to the fact that $\mathbf{v}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r})$ is a subgame perfect feedback Nash equilibrium under the alternative stipulation. In contrast to the remark about the existence assumption (A.3), sufficient conditions for assumptions (A.4) and (A.5) to hold are not known for the general class of differential games under consideration, though for certain differential games, e.g., linear-quadratic, the said differentiability holds.

By assumptions (A.1) through (A.3) and the definition of a feedback Nash equilibrium, the optimal time-path of the pth player's control vector, to wit, $\mathbf{v}^{p}\left(\mathbf{z}(\tau;\boldsymbol{\beta});\boldsymbol{\alpha},\mathbf{r}\right)$, is the solution to the optimal control problem

$$V^{p}(\boldsymbol{\beta})\overset{\text{def}}{=}\max_{\mathbf{u}^{p}(\cdot)}\int_{t}^{+\infty}e^{-r^{p}[\tau-t]}f^{p}\left(\mathbf{x}(\tau),\mathbf{u}^{p}(\tau),\mathbf{v}^{-p}\left(\mathbf{x}(\tau);\boldsymbol{\alpha},\mathbf{r}\right);\boldsymbol{\alpha}\right)dt \tag{2}$$

$$\text{s.t. }\ \dot{\mathbf{x}}(\tau)=\mathbf{g}\Big(\mathbf{x}(\tau),\mathbf{u}^{p}(\tau),\mathbf{v}^{-p}\left(\mathbf{x}(\tau);\boldsymbol{\alpha},\mathbf{r}\right);\boldsymbol{\alpha}\Big),\ \mathbf{x}(t)=\mathbf{x}_{t}\,.$$

The current value Hamiltonian function $H^{p}(\cdot)$ corresponding to optimal control problem (2) is then defined as

$$H^{p}\Big(\mathbf{x},\mathbf{u}^{p},\mathbf{v}^{-p}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r}),\boldsymbol{\lambda}^{p};\boldsymbol{\alpha}\Big)\overset{\text{def}}{=}f^{p}\Big(\mathbf{x},\mathbf{u}^{p},\mathbf{v}^{-p}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r});\boldsymbol{\alpha}\Big)+\boldsymbol{\lambda}^{p\dagger}\mathbf{g}\Big(\mathbf{x},\mathbf{u}^{p},\mathbf{v}^{-p}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r});\boldsymbol{\alpha}\Big). \tag{3}$$

Given that assumptions (A.1) through (A.4) hold, Eqs. (10)–(13) of Starr and Ho (1969a), Theorem 2.2 of Mehlmann (1988), or Theorem 6.15 of Basar and Olsder (1999) give

$$\frac{\partial H^p}{\partial \mathbf{u}^p}\left(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r}),\boldsymbol{\lambda}^p;\boldsymbol{\alpha}\right)=\mathbf{0}^{\dagger}_{M^p}, \tag{4}$$

$$\dot{\boldsymbol{\lambda}}^{p\dagger}=r^p\boldsymbol{\lambda}^{p\dagger}-\frac{\partial H^p}{\partial \mathbf{x}}\left(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r}),\boldsymbol{\lambda}^p;\boldsymbol{\alpha}\right)-\sum_{i=1,i\neq p}^{P}\frac{\partial H^p}{\partial \mathbf{u}^i}\left(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r}),\boldsymbol{\lambda}^p;\boldsymbol{\alpha}\right)\frac{\partial \mathbf{v}^i}{\partial \mathbf{x}}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r}), \tag{5}$$

$$\dot{\mathbf{x}}=\mathbf{g}\left(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r});\boldsymbol{\alpha}\right),\ \mathbf{x}(t)=\mathbf{x}_t, \tag{6}$$

$p\in\{1,2,\ldots,P\}$, as the necessary conditions that a feedback Nash equilibrium $\mathbf{v}\left(\mathbf{z}(\tau;\boldsymbol{\beta});\boldsymbol{\alpha},\mathbf{r}\right)$ must satisfy, along with the associated state-costate pair $\left(\mathbf{z}(\tau;\boldsymbol{\beta}),\boldsymbol{\lambda}(\tau;\boldsymbol{\beta})\right)$, where $\mathbf{0}^{\dagger}_{M^p}$ is the null row vector in $\mathbb{R}^{M^p}$.

It is important to note that the preceding necessary conditions are rarely used to derive or qualitatively characterize a feedback Nash equilibrium of a differential game, even if the mathematical structure of the game is exceedingly simple. As is well known, the difficulty in using Eqs. (4) through (6) to derive a feedback Nash equilibrium resides with Eq. (5), the costate equation. In particular, the appearance of the $M^i\times N$ Jacobian matrix $\partial\mathbf{v}^i(\mathbf{x};\boldsymbol{\alpha},\mathbf{r})/\partial\mathbf{x}$ in the costate equation typically precludes one from fruitfully using the said necessary conditions to derive or qualitatively characterize a feedback Nash equilibrium. Instead, the preferred approach is to make strong functional form assumptions on $f^p(\cdot)$, $p\in\{1,2,\ldots,P\}$, and $\mathbf{g}(\cdot)$, and then employ the Hamilton-Jacobi-Bellman (HJB) equation that corresponds to optimal control problem (2), along with the method of undetermined coefficients—and often some symmetry assumptions—to derive a closed-form solution for the current value function $V^p(\cdot)$ and feedback Nash equilibrium $\mathbf{v}(\cdot)$. Indeed, this is arguably the canonical approach to finding feedback Nash equilibria.

The approach followed below, therefore, is to employ the HJB equation corresponding to optimal control problem (2) to qualitatively characterize feedback Nash equilibria of the class of differential games defined by Eq. (1). Seeing as no functional form assumptions are made regarding the functions $f^p(\cdot)$, $p\in\{1,2,\ldots,P\}$, or $\mathbf{g}(\cdot)$, nor are monotonicity, curvature, or similar assumptions imposed on them, the ensuing qualitative results follow from the basic assumptions

(A.1) through (A.5) and the assertion of optimality alone. Accordingly, the comparative dynamics results derived in §3 are intrinsic to the aforesaid class of differential games.

By Theorem 4.1 of Dockner et al. (2000) and the fact that $V^p(\cdot)$ is not an explicit function of the fixed but arbitrary base time $t \in [0,+\infty)$, the HJB equation corresponding to optimal control problem (2) is given by

$$rV^p(\mathbf{x},\boldsymbol{\alpha},\mathbf{r}) = \max_{\mathbf{u}^p \in \mathbb{R}^{M^p}} \left\{ f^p\left(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r});\boldsymbol{\alpha}\right) + V_{\mathbf{x}}^p(\mathbf{x},\boldsymbol{\alpha},\mathbf{r})\mathbf{g}\left(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r});\boldsymbol{\alpha}\right) \right\} \tag{7}$$

for all $p \in \{1,2,\ldots,P\}$, where $\mathbf{x} \in \mathrm{X}$ is the value of the state vector at the fixed but arbitrary base time $t \in [0,+\infty)$. By assumptions (A.1) through (A.5), $\mathbf{v}^p(\mathbf{x};\boldsymbol{\alpha},\mathbf{r})$ is necessarily a locally differentiable solution on Ω of the maximization problem defined in Eq. (7). Furthermore, as remarked above, if $\mathbf{v}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r})$ is a feedback Nash equilibrium for the class of differential games defined by Eq. (1) for all $\mathbf{x} \in \mathrm{X}$ and $(\boldsymbol{\alpha},\mathbf{r}) \in B\left((\boldsymbol{\alpha}^\circ,\mathbf{r}^\circ);\varepsilon\right)$, then $\mathbf{v}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r})$ is also subgame perfect. This wraps up the technical preliminaries.

The following section states and proves the central result of the paper. It also identifies a set of separability conditions under which the intrinsic comparative dynamics of a differential game collapse, in form, to those in optimal control problems. Moreover, it is shown how the said separability conditions are common to many of the applied differential game models in the literature.

3. Intrinsic Comparative Dynamics

The goal of the present section is to derive the intrinsic comparative dynamics of feedback Nash equilibria for the class of autonomous, exponentially discounted, and infinite horizon differential games defined by Eq. (1). The proof is surprisingly compact given the generic nature of the class of differential games being contemplated. This is due to the use of recent advances in the theory of comparative statics for optimization problems due to Partovi and Caputo (2006, 2007). The central result of the paper is given in the ensuing theorem.

Theorem 1 (Intrinsic comparative dynamics and maximal rank). *Under assumptions (A.1) through (A.5), the intrinsic comparative dynamics of the feedback Nash equilibria* $\mathbf{v}(\boldsymbol{\beta})$ *of differential game (1) are summarized by the statement that the* $(N+K+P)\times(N+K+P)$ *matrix* $\mathbf{S}^p\left(\mathbf{v}(\boldsymbol{\beta})\right)=\left[S^p_{ij}\left(\mathbf{v}(\boldsymbol{\beta})\right)\right]$, $p\in\{1,2,\ldots,P\}$, $i,j=1,2,\ldots,N+K+P$, *is symmetric and positive semidefinite on* Ω, *where*

$$\begin{aligned}\underset{i,j=1,2,\ldots,N}{S^p_{ij}\left(\mathbf{v}(\boldsymbol{\beta})\right)} &= \sum_{m=1}^{M^p}\left[f^p_{x_i u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)+\sum_{n=1}^{N}\left[V^p_{x_n}(\boldsymbol{\beta})g^n_{x_i u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)+V^p_{x_n x_i}(\boldsymbol{\beta})g^n_{u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)\right]\right]\frac{\partial v^p_m(\boldsymbol{\beta})}{\partial x_j}\\ &+\sum_{m=1}^{M^p}\sum_{\substack{q=1\\q\neq p}}^{P}\sum_{s=1}^{M^q}\left[f^p_{u^q_s u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)+\sum_{n=1}^{N}\left[V^p_{x_n}(\boldsymbol{\beta})g^n_{u^q_s u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)\right]\right]\frac{\partial v^q_s(\boldsymbol{\beta})}{\partial x_i}\frac{\partial v^p_m(\boldsymbol{\beta})}{\partial x_j},\end{aligned}\tag{8}$$

$$\begin{aligned}\underset{\substack{i=1,2,\ldots,N\\ j=N+1,\ldots,N+K}}{S^p_{ij}\left(\mathbf{v}(\boldsymbol{\beta})\right)} &= \sum_{m=1}^{M^p}\left[f^p_{x_i u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)+\sum_{n=1}^{N}\left[V^p_{x_n}(\boldsymbol{\beta})g^n_{x_i u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)+V^p_{x_n x_i}(\boldsymbol{\beta})g^n_{u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)\right]\right]\frac{\partial v^p_m(\boldsymbol{\beta})}{\partial \alpha_{j-N}}\\ &+\sum_{m=1}^{M^p}\sum_{\substack{q=1\\q\neq p}}^{P}\sum_{s=1}^{M^q}\left[f^p_{u^q_s u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)+\sum_{n=1}^{N}\left[V^p_{x_n}(\boldsymbol{\beta})g^n_{u^q_s u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)\right]\right]\frac{\partial v^q_s(\boldsymbol{\beta})}{\partial x_i}\frac{\partial v^p_m(\boldsymbol{\beta})}{\partial \alpha_{j-N}},\end{aligned}\tag{9}$$

$$\begin{aligned}\underset{\substack{i=1,2,\ldots,N\\ j=N+K+1,\ldots,N+K+P}}{S^p_{ij}\left(\mathbf{v}(\boldsymbol{\beta})\right)} &= \sum_{m=1}^{M^p}\left[f^p_{x_i u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)+\sum_{n=1}^{N}\left[V^p_{x_n}(\boldsymbol{\beta})g^n_{x_i u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)+V^p_{x_n x_i}(\boldsymbol{\beta})g^n_{u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)\right]\right]\frac{\partial v^p_m(\boldsymbol{\beta})}{\partial r^{j-N-K}}\\ &+\sum_{m=1}^{M^p}\sum_{\substack{q=1\\q\neq p}}^{P}\sum_{s=1}^{M^q}\left[f^p_{u^q_s u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)+\sum_{n=1}^{N}\left[V^p_{x_n}(\boldsymbol{\beta})g^n_{u^q_s u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)\right]\right]\frac{\partial v^q_s(\boldsymbol{\beta})}{\partial x_i}\frac{\partial v^p_m(\boldsymbol{\beta})}{\partial r^{j-N-K}},\end{aligned}\tag{10}$$

$$\begin{aligned}\underset{\substack{i=N+1,\ldots,N+K\\ j=1,2,\ldots,N}}{S^p_{ij}\left(\mathbf{v}(\boldsymbol{\beta})\right)} &= \sum_{m=1}^{M^p}\left[f^p_{\alpha_{i-N} u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)+\sum_{n=1}^{N}\left[V^p_{x_n}(\boldsymbol{\beta})g^n_{\alpha_{i-N} u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)+V^p_{x_n \alpha_{i-N}}(\boldsymbol{\beta})g^n_{u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)\right]\right]\frac{\partial v^p_m(\boldsymbol{\beta})}{\partial x_j}\\ &+\sum_{m=1}^{M^p}\sum_{\substack{q=1\\q\neq p}}^{P}\sum_{s=1}^{M^q}\left[f^p_{u^q_s u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)+\sum_{n=1}^{N}\left[V^p_{x_n}(\boldsymbol{\beta})g^n_{u^q_s u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)\right]\right]\frac{\partial v^q_s(\boldsymbol{\beta})}{\partial \alpha_{i-N}}\frac{\partial v^p_m(\boldsymbol{\beta})}{\partial x_j},\end{aligned}\tag{11}$$

$$\begin{aligned}\underset{i,j=N+1,\ldots,N+K}{S^p_{ij}\left(\mathbf{v}(\boldsymbol{\beta})\right)} &= \sum_{m=1}^{M^p}\left[f^p_{\alpha_{i-N} u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)+\sum_{n=1}^{N}\left[V^p_{x_n}(\boldsymbol{\beta})g^n_{\alpha_{i-N} u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)+V^p_{x_n \alpha_{i-N}}(\boldsymbol{\beta})g^n_{u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)\right]\right]\frac{\partial v^p_m(\boldsymbol{\beta})}{\partial \alpha_{j-N}}\\ &+\sum_{m=1}^{M^p}\sum_{\substack{q=1\\q\neq p}}^{P}\sum_{s=1}^{M^q}\left[f^p_{u^q_s u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)+\sum_{n=1}^{N}\left[V^p_{x_n}(\boldsymbol{\beta})g^n_{u^q_s u^p_m}\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)\right]\right]\frac{\partial v^q_s(\boldsymbol{\beta})}{\partial \alpha_{i-N}}\frac{\partial v^p_m(\boldsymbol{\beta})}{\partial \alpha_{j-N}},\end{aligned}\tag{12}$$

$$\underset{\substack{i=N+1,\ldots,N+K\\ j=N+K+1,\ldots,N+K+P}}{S_{ij}^{p}\left(\mathbf{v}(\boldsymbol{\beta})\right)} = \sum_{m=1}^{M^p}\left[f_{\alpha_{i-N}u_m^p}^{p}(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}) + \sum_{n=1}^{N}\left[V_{x_n}^{p}(\boldsymbol{\beta})g_{\alpha_{i-N}u_m^p}^{n}(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}) + V_{x_n\alpha_{i-N}}^{p}(\boldsymbol{\beta})g_{u_m^p}^{n}(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha})\right]\right]\frac{\partial v_m^p(\boldsymbol{\beta})}{\partial r^{j-N-K}}$$
$$+\sum_{m=1}^{M^p}\sum_{\substack{q=1\\ q\neq p}}^{P}\sum_{s=1}^{M^q}\left[f_{u_s^q u_m^p}^{p}(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}) + \sum_{n=1}^{N}\left[V_{x_n}^{p}(\boldsymbol{\beta})g_{u_s^q u_m^p}^{n}(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha})\right]\right]\frac{\partial v_s^q(\boldsymbol{\beta})}{\partial \alpha_{i-N}}\frac{\partial v_m^p(\boldsymbol{\beta})}{\partial r^{j-N-K}}, \tag{13}$$

$$\underset{\substack{i=N+K+1,\ldots,N+K+P\\ j=1,2,\ldots,N}}{S_{ij}^{p}\left(\mathbf{v}(\boldsymbol{\beta})\right)} = \sum_{m=1}^{M^p}\sum_{n=1}^{N}V_{x_n r^{i-N-K}}^{p}(\boldsymbol{\beta})g_{u_m^p}^{n}(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha})\frac{\partial v_m^p(\boldsymbol{\beta})}{\partial x_j}$$
$$+\sum_{m=1}^{M^p}\sum_{\substack{q=1\\ q\neq p}}^{P}\sum_{s=1}^{M^q}\left[f_{u_s^q u_m^p}^{p}(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}) + \sum_{n=1}^{N}\left[V_{x_n}^{p}(\boldsymbol{\beta})g_{u_s^q u_m^p}^{n}(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha})\right]\right]\frac{\partial v_s^q(\boldsymbol{\beta})}{\partial r^{i-N-K}}\frac{\partial v_m^p(\boldsymbol{\beta})}{\partial x_j}, \tag{14}$$

$$\underset{\substack{i=N+K+1,\ldots,N+K+P\\ j=N+1,\ldots,N+K}}{S_{ij}^{p}\left(\mathbf{v}(\boldsymbol{\beta})\right)} = \sum_{m=1}^{M^p}\sum_{n=1}^{N}V_{x_n r^{i-N-K}}^{p}(\boldsymbol{\beta})g_{u_m^p}^{n}(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha})\frac{\partial v_m^p(\boldsymbol{\beta})}{\partial \alpha_{j-N}}$$
$$+\sum_{m=1}^{M^p}\sum_{\substack{q=1\\ q\neq p}}^{P}\sum_{s=1}^{M^q}\left[f_{u_s^q u_m^p}^{p}(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}) + \sum_{n=1}^{N}\left[V_{x_n}^{p}(\boldsymbol{\beta})g_{u_s^q u_m^p}^{n}(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha})\right]\right]\frac{\partial v_s^q(\boldsymbol{\beta})}{\partial r^{i-N-K}}\frac{\partial v_m^p(\boldsymbol{\beta})}{\partial \alpha_{j-N}}, \tag{15}$$

$$\underset{i,j=N+K+1,\ldots,N+K+P}{S_{ij}^{p}\left(\mathbf{v}(\boldsymbol{\beta})\right)} = \sum_{m=1}^{M^p}\sum_{n=1}^{N}V_{x_n r^{i-N-K}}^{p}(\boldsymbol{\beta})g_{u_m^p}^{n}(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha})\frac{\partial v_m^p(\boldsymbol{\beta})}{\partial r^{j-N-K}}$$
$$+\sum_{m=1}^{M^p}\sum_{\substack{q=1\\ q\neq p}}^{P}\sum_{s=1}^{M^q}\left[f_{u_s^q u_m^p}^{p}(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}) + \sum_{n=1}^{N}\left[V_{x_n}^{p}(\boldsymbol{\beta})g_{u_s^q u_m^p}^{n}(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha})\right]\right]\frac{\partial v_s^q(\boldsymbol{\beta})}{\partial r^{i-N-K}}\frac{\partial v_m^p(\boldsymbol{\beta})}{\partial r^{j-N-K}}, \tag{16}$$

Furthermore, $\operatorname{rank}\left(\mathbf{S}^p\left(\mathbf{v}(\boldsymbol{\beta})\right)\right) \le \min(M^p, N+K+P)$, $p \in \{1,2,\ldots,P\}$.

Proof. Define the maximand on the right-hand side of the H-J-B equation given in Eq. (2) as

$$F^p(\mathbf{u}^p;\boldsymbol{\beta}) \overset{\text{def}}{=} f^p\left(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r});\boldsymbol{\alpha}\right) + \sum_{n=1}^{N}V_{x_n}^{p}(\mathbf{x},\boldsymbol{\alpha},\mathbf{r})g^n\left(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r});\boldsymbol{\alpha}\right). \tag{17}$$

Observe that $F^p(\mathbf{u}^p;\boldsymbol{\beta})$ is the Hamiltonian for optimal control problem (2) with the costate vector replaced by the gradient vector of the current value optimal value function of player p with respect to the state vector. By Theorem 1 of Partovi and Caputo (2006, 2007), it follows that for all $p \in \{1,2,\ldots,P\}$, the $(N+K+P)\times(N+K+P)$ matrix

$$\mathbf{S}^p\left(\mathbf{v}(\boldsymbol{\beta})\right) = \left[\underset{i,j=1,2,\ldots,N+K+P}{S_{ij}^{p}\left(\mathbf{v}(\boldsymbol{\beta})\right)}\right] \overset{\text{def}}{=} \left[\underset{i,j=1,2,\ldots,N+K+P}{\sum_{m=1}^{M^p}F_{\beta_i u_m^p}^{p}\left(\mathbf{v}^p(\boldsymbol{\beta});\boldsymbol{\beta}\right)\frac{\partial v_m^p(\boldsymbol{\beta})}{\partial \beta_j}}\right], \tag{18}$$

is symmetric and positive semidefinite on Ω in view of assumptions (A.1) through (A.5), and

$$\beta_i \overset{\text{def}}{=} \begin{cases} x_i & \text{if } i = 1,2,\ldots,N, \\ \alpha_{i-N} & \text{if } i = N+1, N+2,\ldots,N+K, \\ r^{i-N-K} & \text{if } i = N+K+1, N+K+2,\ldots,N+K+P. \end{cases} \tag{19}$$

To prove Theorem 2, it must be shown that Eqs. (8) through (16) follow from Eqs. (17) through (19), respectively.

In order to derive, say, Eq. (9), begin by differentiating $F^p(\cdot)$ with respect to β_i for $i = 1,2,\ldots,N$ using Eqs. (17) and (19), and then again with respect to u_m^p to get

$$\begin{aligned} F_{\beta_i}^p(\mathbf{u}^p;\boldsymbol{\beta}) &= f_{x_i}^p\left(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(\boldsymbol{\beta});\boldsymbol{\alpha}\right) + \sum_{n=1}^{N}\left[V_{x_n}^p(\boldsymbol{\beta})g_{x_i}^n\left(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(\boldsymbol{\beta});\boldsymbol{\alpha}\right) + V_{x_n x_i}^p(\boldsymbol{\beta})g^n\left(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)\right] \\ &\quad + \sum_{q=1,q\neq p}^{P}\sum_{s=1}^{M^q}\left[f_{u_s^q}^p\left(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(\boldsymbol{\beta});\boldsymbol{\alpha}\right) + \sum_{n=1}^{N}\left[V_{x_n}^p(\boldsymbol{\beta})g_{u_s^q}^n\left(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)\right]\right]\frac{\partial v_s^q(\boldsymbol{\beta})}{\partial x_i}, \\ F_{\beta_i u_m^p}^p\left(\mathbf{v}^p(\boldsymbol{\beta});\boldsymbol{\beta}\right) &= f_{x_i u_m^p}^p\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right) + \sum_{n=1}^{N}\left[V_{x_n}^p(\boldsymbol{\beta})g_{x_i u_m^p}^n\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right) + V_{x_n x_i}^p(\boldsymbol{\beta})g_{u_m^p}^n\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)\right] \\ &\quad + \sum_{\substack{q=1\\q\neq p}}^{P}\sum_{s=1}^{M^q}\left[f_{u_s^q u_m^p}^p\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right) + \sum_{n=1}^{N}\left[V_{x_n}^p(\boldsymbol{\beta})g_{u_s^q u_m^p}^n\left(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\right)\right]\right]\frac{\partial v_s^q(\boldsymbol{\beta})}{\partial x_i}, \end{aligned} \tag{20}$$

where Eq. (20) has been evaluated at $\mathbf{u}^p = \mathbf{v}^p(\boldsymbol{\beta})$, thereby implying that the vector consisting of all the players' control vectors has the value $\mathbf{v}(\boldsymbol{\beta})$. Upon multiplying Eq. (20) by the term $\partial v_m^p(\boldsymbol{\beta})/\partial\beta_j = \partial v_m^p(\boldsymbol{\beta})/\partial\alpha_{j-N}$ for $j = N+1, N+2,\ldots,N+K$ noting Eq. (19), and then summing the resulting expression over m, from $m=1$ to $m = M^p$, all of which are dictated by Eq. (18), yields Eq. (9). The proof for the remaining eight submatrices of $\mathbf{S}^p\left(\mathbf{v}(\boldsymbol{\beta})\right)$ follows the same pattern and is therefore left for the reader. Applying Theorem 4 of Partovi and Caputo (2006) yields the rank conclusion. *Q.E.D.*

At this juncture it worthwhile to pause and make seven extended remarks on Theorem 1. First note that the comparative dynamics given in Theorem 1 are heretofore unknown. What is more, they are intrinsic to the class of autonomous, exponentially discounted, infinite horizon differential games defined by Eq. (1), seeing as they follow solely from the assertion of maximization and the basic assumptions (A.1) through (A.5), and are thus are not predicated on functional form suppositions or ad hoc qualitative properties imposed on the instantaneous payoff and transition functions. As a result, the qualitative results in Theorem 1 form the basic, testable implications of *all* differential games of the aforesaid class.

Second, recall that if $\mathbf{v}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r})$ is a feedback Nash equilibrium for the class of differential games defined by Eq. (1) for all $\mathbf{x}\in \mathrm{X}$ and for all $(\boldsymbol{\alpha},\mathbf{r})\in B\big((\boldsymbol{\alpha}^{\circ},\mathbf{r}^{\circ});\varepsilon\big)$, then $\mathbf{v}(\mathbf{x};\boldsymbol{\alpha},\mathbf{r})$ is also subgame perfect. In this case, if assumptions (A.4) and (A.5) are extended to hold for all $\mathbf{x}\in \mathrm{X}$ and for all $(\boldsymbol{\alpha},\mathbf{r})\in B\big((\boldsymbol{\alpha}^{\circ},\mathbf{r}^{\circ});\varepsilon\big)$, then Theorem 1 also characterizes the intrinsic comparative dynamics of subgame perfect feedback Nash equilibria. Consequently, it follows that the intrinsic comparative dynamics of feedback Nash equilibria and subgame perfect feedback Nash equilibria for the class of differential games defined by Eq. (1) are qualitatively identical. It is worthwhile to observe that the elements of the symmetric and positive semidefinite matrix $\mathbf{S}^{p}\big(\mathbf{v}(\boldsymbol{\beta})\big)$ may differ in magnitude between the two equilibria, however, as the equilibria may differ themselves—see, e.g., Dockner et al. (2000), Example 4.2 and Example 4.2 (continued).

Third, as asserted in §1, the form that the basic comparative dynamics take has, in part, the flavor of a generalized Slutsky-like expression. Consider, for example, Eq. (8). The generalized Slutsky-like expression is the first term in Eq. (8), seeing as it consists of a linear combination of the partial derivatives of the *p*th player's feedback Nash equilibrium control variables with respect to the base time values of the state variables, to wit,

$$\sum_{m=1}^{M^p}\left[f^p_{x_i u^p_m}\big(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\big)+\sum_{n=1}^{N}\left[V^p_{x_n}(\boldsymbol{\beta})g^n_{x_i u^p_m}\big(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\big)+V^p_{x_n x_i}(\boldsymbol{\beta})g^n_{u^p_m}\big(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\big)\right]\right]\frac{\partial v^p_m(\boldsymbol{\beta})}{\partial x_j}. \tag{21}$$

The second term in Eq. (8), videlicet,

$$\sum_{m=1}^{M^p}\sum_{\substack{q=1\\ q\neq p}}^{P}\sum_{s=1}^{M^q}\left[f^p_{u^q_s u^p_m}\big(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\big)+\sum_{n=1}^{N}\left[V^p_{x_n}(\boldsymbol{\beta})g^n_{u^q_s u^p_m}\big(\mathbf{x},\mathbf{v}(\boldsymbol{\beta});\boldsymbol{\alpha}\big)\right]\right]\frac{\partial v^q_s(\boldsymbol{\beta})}{\partial x_i}\frac{\partial v^p_m(\boldsymbol{\beta})}{\partial x_j}, \tag{22}$$

does not have a Slutsky-like flavor, as it includes the effect that a change in the base time value of a state variable has on the other players' feedback Nash equilibrium value of their control variables, i.e., $\partial v^q_s(\boldsymbol{\beta})/\partial x_i$. Such a term does not have an analogue in the Slutsky equation. The expression appearing in Eq. (22) is fundamentally what distinguishes the intrinsic comparative dynamics of feedback Nash equilibria of differential games from those of feedback (or closed-loop) equilibria of optimal control problems, as will be shown below. Because Eq. (22) includes the term $\partial v_s^q(\boldsymbol{\beta})/\partial x_i$, it captures the impact that the strategic nature of the differential game has on its intrinsic comparative dynamics. It is therefore appropriate and natural to define Eq. (21) and its counterparts in Eqs. (9) through (16) as the *nonstrategic comparative dynamics effect*, and Eq. (22) and its counterparts in Eqs. (9) through (16) as the *strategic comparative dynamics effect*.

Fourth, the fact that the rank of the symmetric and positive semidefinite matrix $\mathbf{S}^p(\mathbf{v}(\boldsymbol{\beta}))$ cannot exceed the smaller of (i) the number of control variables of player p, or (ii) the sum of the number of state variables, parameters, and discounts rates in the differential game, implies that for most of the applied differential games appearing in the economics literature, $\mathbf{S}^p(\mathbf{v}(\boldsymbol{\beta}))$ is singular. For example, in the capital accumulation game developed in Spence (1979) and Dockner et al. (2000, p. 244). each firm has one control variable, namely, its gross rate of investment, and there is also one firm-specific state variable for each firm, viz., its capital stock, for P in total. Moreover, each capital accumulation equation has a firm-specific rate of depreciation and the constant price per unit of investment is common to each firm, yielding $P+1$ parameters (exclusive of the discount rate) in the game. There is also a common rate of discount employed by the firms. As a result, it follows that $M^p = 1$ and $N+K+P = P+[P+1]+1 = 2[P+1]$ in the game. Hence $\mathbf{S}^p(\mathbf{v}(\boldsymbol{\beta}))$ is singular, as its order, $2[P+1]$, exceeds its maximum rank, unity, by at least $2P+1$. Another differential game in which $\mathbf{S}^p(\mathbf{v}(\boldsymbol{\beta}))$ is singular is the sticky price duopoly game of Fershtman and Kamien (1987). In this game each firm has one control variable that influences the evolution of a common state variable. In addition, the firms use the same discount rate and face three other parameters, hence $M^p = 1$ and $N+K+P = 1+3+1 = 5$, thereby implying that the order of $\mathbf{S}^p(\mathbf{v}(\boldsymbol{\beta}))$ exceeds its rank by at least four.

More generally, the singularity of $\mathbf{S}^p(\mathbf{v}(\boldsymbol{\beta}))$ holds for the class of differential games in which each player has a single control variable and there is only a single state variable in the game, because even if no other parameters appear in it and the players use a common discount rate, $M^p = 1$ and $N + K + P = 1 + 0 + 1 = 2$, thereby implying that $\mathbf{S}^p(\mathbf{v}(\boldsymbol{\beta}))$ is singular. It is not a stretch, therefore, to claim that for most of the applied differential games in the literature, that $\mathbf{S}^p(\mathbf{v}(\boldsymbol{\beta}))$ is singular. This is especially so for games in which a closed-form solution for a feedback Nash equilibrium is derived, as each player typically has but one control variable in such games. By Theorem 1, the rank of the intrinsic comparative dynamics matrix of such games cannot exceed unity. The other fact worth mentioning is that the rank of $\mathbf{S}^p(\mathbf{v}(\boldsymbol{\beta}))$ may be reduced below the upper bound given in Theorem 1 due to the specific structure of the basic functions in the game, say by an assumed homogeneity property of a production function.

Fifth, the intrinsic comparative statics of Nash equilibria for unconstrained static games given by Corollary 2 of Caputo (1996) are a special case of Theorem 1. To see this, begin by noting that in a static game, the decision variables are the analogues of the control variables in a differential game. Because dynamics are absent in a static game, the vector of transition functions is identically zero, i.e., $\mathbf{g}(\cdot) \equiv \mathbf{0}_N$. Moreover, the vector of state variables $\mathbf{x}$ is absent in a static game, as is the vector of discount rates $\mathbf{r}$, leaving only the vector of parameters $\boldsymbol{\alpha}$ in a static game. Consequently, by setting $\mathbf{g}(\cdot) \equiv \mathbf{0}_N$ in Eq. (12) and noting the aforementioned facts, it is seen that the resulting expression is identical to Corollary 2 of Caputo (1996).

Sixth, the intrinsic comparative dynamics of feedback (or closed-loop) optimal controls for the class of autonomous and exponentially discounted infinite horizon optimal control problems given by Theorem 2 of Caputo (2003) are also a special case of Theorem 1. To see this, recall that in an optimal control problem there is but one player, i.e., $p = P = 1$, and thus one discount rate r, thereby implying that $\boldsymbol{\beta} \overset{\text{def}}{=} (\mathbf{x}^\dagger, \boldsymbol{\alpha}^\dagger, r)^\dagger \in X \times \mathbb{R}^A \times \mathbb{R}_{++}$. These facts imply that $\partial \mathbf{v}^q(\boldsymbol{\beta}) / \partial \boldsymbol{\beta} \equiv \mathbf{0}_{M^q \times (N+K+1)}$ for $q \neq p$. As a result, the strategic comparative dynamics effect in each of the nine submatrices in Eqs. (8) through (16) of Theorem 1, that is, the term involving the triple summation over m, s, and q, vanishes identically, leaving only the nonstrategic term in each submatrix. But these nonstrategic terms constitute exactly the nine submatrices of Theorem 2 of Caputo (2003), thereby establishing the claim.

And seventh, if (i) the instantaneous payoff function of every player is additively separable between that player's control variables and those of every other player, and (ii) all the transition functions are additively separable with respect to the control variables of the different players, then the strategic comparative dynamics effect of feedback Nash equilibria vanishes. The above form of additive separability may be stated in the ensuing mathematical form: $f^p_{u^q_s u^p_m}(\mathbf{x},\mathbf{u}^p,\mathbf{u}^{-p};\boldsymbol{\alpha}) \equiv 0$ and $g^n_{u^q_s u^p_m}(\mathbf{x},\mathbf{u}^p,\mathbf{u}^{-p};\boldsymbol{\alpha}) \equiv 0$ for $p,q \in \{1,2,\ldots,P\}$, $q \neq p$, $m = 1,2,\ldots,M^p$, $n = 1,2,\ldots,N$, and $s = 1,2,\ldots,M^q$. Using these conditions in Theorem 1, it follows that the term involving the triple summation over m, s, and q vanishes identically in Eqs. (8) through (16), i.e., the strategic comparative dynamics effects vanish identically, leaving only the nonstrategic effect. This result is sufficiently novel, and the separability conditions are quite common, as indicated below, that it is worthwhile to have a formal statement of the result.

Corollary 1. *If* $f^p_{u^q_s u^p_m}(\mathbf{x},\mathbf{u}^p,\mathbf{u}^{-p};\boldsymbol{\alpha}) \equiv 0$ *and* $g^n_{u^q_s u^p_m}(\mathbf{x},\mathbf{u}^p,\mathbf{u}^{-p};\boldsymbol{\alpha}) \equiv 0$ *for* $p,q \in \{1,2,\ldots,P\}$, $q \neq p$, $m = 1,2,\ldots,M^p$, $n = 1,2,\ldots,N$, *and* $s = 1,2,\ldots,M^q$, *then the intrinsic comparative dynamics of feedback Nash equilibria for the class of differential games defined by Eq. (1) do not contain any strategic comparative dynamics effect.*

We close this section with three remarks on Corollary 1. First, observe that the additive separability of a given player's instantaneous payoff function does not (i) apply to that player's control variables, (ii) pertain to the control variables of any two other players, nor (iii) involve any of the parameters or state variables. Moreover, the said separability on the transitions functions does not (i) apply to a given player's control variables, nor (ii) involve any of the parameters or state variables. Other related, but quite different, separability restrictions have proven to be of some value in the differential game literature. For example, Dockner et al. (1985) and Dockner et al. (2000, Chapter 7) identify structural assumptions on the instantaneous payoff and transition functions of the players that lead to an analytically solvable class of differential games. In particular, they show that linearity in the state variables and additive separability between all of the control and state variables of the aforesaid functions yields an analytically solvable class of differential games with an open-loop information structure. Similarly, Bacchiega et al. (2010), building off of Rubio (2006), employ separability assumptions that apply to two groups of mutually exclusive players in a class of differential games in which each player has one control variable and the number of state variables equals the number of players. Specifically, they show that if the integrand and transition functions of the said class of differential game are additively separable with respect to the state and control variables of players in the two mutually exclusive groups, then feedback Nash and feedback Stackelberg equilibria coincide.

Second, the additive separability does not have to hold globally, even though the corollary states it in such terms. Indeed, as long as the additive separability holds at a feedback Nash equilibrium, Corollary 1 still holds. Nonetheless, it is easier to check that the separability holds globally—it can be done simply by inspection of the instantaneous payoff and transition functions—than if it holds at a feedback Nash equilibrium, because in the latter case, the feedback Nash equilibrium would have to be computed, typically a task only suitable for tightly specified instantaneous payoff and transition functions.

And third, Corollary 1 applies to many of the applied differential game models in the literature. A sample of differential games for which Corollary 1 is applicable includes the sticky price games of Fershtman and Kamien (1987) and Cellini and Lambertini (2007), the oligopoly pricing games in Dockner (1984) and Feichtinger and Dockner (1985), the Vidale-Wolfe-, Lanchester- and Leitmann-type advertising games surveyed by Jørgensen (1982) and the modified Case-type advertising game in Sorger (1989), the international pollution control game in Dockner and Long (1993) as well as many of the environmental games surveyed by Long (2010, Chapter 2), the common property resource games in Negri (1989), Rubio and Casino (2001), along with many of the resource games surveyed in Long (2010, Chapter 3) and Long (2011), the capital accumulation games in Spence (1979), Reynolds (1987, 1991), Dockner et al. (2000, Chapter 9), Jun and Vives (2004), and Figuières (2009), as well as the industrial organization games surveyed by Long (2010, Chapter 5), and finally, the workhorse (and essentially ubiquitous) class of linear-quadratic differential games in Mehlmann (1988, Chapter 4), Dockner et al. (2000, Chapter 7), and Engwerda (2005, Chapter 8).

4. A Capital Accumulation Game

The differential game under consideration in this section is a generalization of the finite horizon capital accumulation game of Spence (1979). See Dockner et al. (2000, Chapter 9) and Long (2010, Chapter 5) for a comprehensive overview of the literature stemming from the seminal paper of Spence (1979). In the more general case studied here, the differential game takes the form

$$V^p(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{I^p(\cdot)} \int_t^{+\infty} \left[\pi^p\left(\mathbf{K}(\tau)\right) - w^p K^p(\tau) - qI^p(\tau) - C^p\left(I^p(\tau)\right)\right] e^{-r[\tau - t]} d\tau \tag{23}$$

$$\text{s.t. } \dot{K}^n(\tau) = I^n(\tau) - \delta^n K^n(\tau),\ K^n(t) = K_t^n,\ n = 1,2,\ldots,P,$$

where $p \in \{1,2,\ldots,P\}$ indexes the firms, $K^p(\tau) > 0$ is the pth firm's capital stock, $w^p > 0$ is the pth firm's maintenance cost per unit of capital, $I^p(\tau)$ is the pth firm's reversible investment rate and thus its control variable, $q > 0$ is the price per unit of investment faced by all the firms, $\pi^p(\mathbf{K})$ is the pth firm's maximum profit flow net of variable production costs, $C^p(\cdot)$ is the adjustment cost function of the pth firm, $r > 0$ is the common rate of discount used by the firms, and $\delta^p > 0$ is the depreciation rate of the pth firm's capital stock. It is also useful to define $\mathbf{K}(\tau) \stackrel{\text{def}}{=} \left(K^1(\tau), K^2(\tau), \ldots, K^P(\tau)\right)^\dagger \in \mathbb{R}_{++}^P$, $\mathbf{w} \stackrel{\text{def}}{=} (w^1, w^2, \ldots, w^P)^\dagger \in \mathbb{R}_{++}^P$, $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (q, \mathbf{w}^\dagger)^\dagger \in \mathbb{R}_{++}^{P+1}$, and $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\mathbf{K}_t^\dagger, \boldsymbol{\alpha}^\dagger, r)^\dagger \in \mathbb{R}_{++}^{2P+2}$. Note that the rates of depreciation have been suppressed from $\boldsymbol{\alpha}$, as the effects of changes in the δ^n on a feedback Nash equilibrium are not considered so as to keep the ensuing exposition relatively free from an excessive number of equations. Denote a feedback Nash equilibrium for the differential game by $\mathbf{I}^*(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \left(I^{1*}(\boldsymbol{\beta}), I^{2*}(\boldsymbol{\beta}), \ldots, I^{P*}(\boldsymbol{\beta})\right)^\dagger \in \mathbb{R}^P$. Finally, suppositions (A.1) through (A.5) are assumed to hold in what follows.

An inspection of the capital accumulation game in Eq. (23) shows that the separability conditions in Corollary 1 hold, thereby implying that the strategic comparative dynamics effects vanish. Accordingly, applying Theorem 1 to the above game leads to a rather simple form for the $(2P+2) \times (2P+2)$ symmetric and positive semidefinite matrix $\mathbf{S}^p\left(\mathbf{I}^*(\boldsymbol{\beta})\right) = \left[S_{ij}^p\left(\mathbf{I}^*(\boldsymbol{\beta})\right)\right]$, $p \in \{1,2,\ldots,P\}$, $i, j = 1,2,\ldots,2P+1$, namely

$$\mathbf{S}^p\left(\mathbf{I}^*(\boldsymbol{\beta})\right)=\left[\begin{array}{c:c:c:c} \underset{i,j=1,2,\ldots,P}{V^p_{K^pK^i}\dfrac{\partial I^{p*}}{\partial K^j}} & \underset{\substack{i=1,2,\ldots,P\\ j=P+1}}{V^p_{K^pK^i}\dfrac{\partial I^{p*}}{\partial q}} & \underset{\substack{i,j=1,2,\ldots,P\\ j=P+2,P+3,\ldots 2P+1}}{V^p_{K^pK^i}\dfrac{\partial I^{p*}}{\partial w^{j-P-1}}} & \underset{\substack{i=1,2,\ldots,P\\ j=2P+2}}{V^p_{K^pK^i}\dfrac{\partial I^{p*}}{\partial r}} \\ \hdashline \underset{\substack{i=P+1\\ j=1,2,\ldots,P}}{[V^p_{K^pq}-1]\dfrac{\partial I^{p*}}{\partial K^j}} & \underset{i,j=P+1}{[V^p_{K^pq}-1]\dfrac{\partial I^{p*}}{\partial q}} & \underset{\substack{i=P+1\\ j=P+2,P+3,\ldots 2P+1}}{[V^p_{K^pq}-1]\dfrac{\partial I^{p*}}{\partial w^{j-P-1}}} & \underset{\substack{i=P+1\\ j=2P+2}}{[V^p_{K^pq}-1]\dfrac{\partial I^{p*}}{\partial r}} \\ \hdashline \underset{\substack{i=P+2,P+3,\ldots 2P+1\\ j=1,2,\ldots,P}}{V^p_{K^pw^{i-P-1}}\dfrac{\partial I^{p*}}{\partial K^j}} & \underset{\substack{i=P+2,P+3,\ldots 2P+1\\ j=P+1}}{V^p_{K^pw^{i-P-1}}\dfrac{\partial I^{p*}}{\partial q}} & \underset{i,j=P+2,P+3,\ldots 2P+1}{V^p_{K^pw^{i-P-1}}\dfrac{\partial I^{p*}}{\partial w^{j-P-1}}} & \underset{\substack{i=P+2,P+3,\ldots 2P+1\\ j=2P+2}}{V^p_{K^pw^{i-P-1}}\dfrac{\partial I^{p*}}{\partial r}} \\ \hdashline \underset{\substack{i=2P+2\\ j=1,2,\ldots,P}}{V^p_{K^pr}\dfrac{\partial I^{p*}}{\partial K^j}} & \underset{\substack{i=2P+2\\ j=P+1}}{V^p_{K^pr}\dfrac{\partial I^{p*}}{\partial q}} & \underset{\substack{i=2P+2\\ j=P+2,P+3,\ldots,2P+1}}{V^p_{K^pr}\dfrac{\partial I^{p*}}{\partial w^{j-P-1}}} & \underset{i,j=2P+2}{V^p_{K^pr}\dfrac{\partial I^{p*}}{\partial r}} \end{array}\right], \qquad (24)$$

where the parameter vector $\boldsymbol{\beta}$ has been suppressed. Furthermore, Theorem 1 implies that $\operatorname{rank}\left(\mathbf{S}^p\left(\mathbf{I}^*(\boldsymbol{\beta})\right)\right)\leq \min(M^p,N+K+1)=\min(1,P+[P+1]+1)=1$. The symmetry and positive semidefiniteness of $\mathbf{S}^p\left(\mathbf{I}^*(\boldsymbol{\beta})\right)$ for each $p\in\{1,2,\ldots,P\}$, along with its maximal rank, represents the heretofore undiscovered intrinsic qualitative properties of the capital accumulation game defined by Eq. (23).

One noteworthy feature of $\mathbf{S}^p\left(\mathbf{I}^*(\boldsymbol{\beta})\right)$ is its form. Each element essentially consists of the product of two terms, scilicet, (i) the effect of a change in a capital stock or price on the current value shadow price of the pth firm's capital stock, e.g., $V^p_{K^pK^i}(\boldsymbol{\beta})$, and (ii) the effect of a change in a capital stock or price on the pth firm's rate of investment, say, $\partial I^{p*}(\boldsymbol{\beta})/\partial q$. This implies that in order to carry out an empirical test of the intrinsic comparative dynamics of the capital accumulation game, one must estimate a firm's investment demand function $I^{p*}(\cdot)$ and its current value function $V^p(\cdot)$. This differs sharply from the price-taking profit maximizing model of the firm, as one can estimate the indirect profit function or the factor demand and output supply functions in order to test its intrinsic comparative statics. That is, estimation of both sets of functions is not required in order to test its intrinsic comparative statics, though one might wish to estimate both for the econometric reasons, e.g., improved efficiency of the coefficient estimates. The form of $\mathbf{S}^p\left(\mathbf{I}^*(\boldsymbol{\beta})\right)$ demonstrates that the basic comparative dynamics of the capital accumulation game do not consist of the signs of the individual partial derivatives of the investment demand functions with respect to the capital stocks and prices. This also stands in contrast to the basic comparative statics of the aforementioned profit maximization model. Nevertheless, the symmetry and positive semidefiniteness of $\mathbf{S}^p\left(\mathbf{I}^*(\boldsymbol{\beta})\right)$ is a fundamental and empirically testable property of the game.

The positive semidefiniteness of $\mathbf{S}^p\left(\mathbf{I}^*(\boldsymbol{\beta})\right)$ implies that its elements along the main diagonal are nonnegative, thus yielding for each $p \in \{1,2,\ldots,P\}$,

$$\operatorname{sign}\left[\frac{\partial I^{p*}(\boldsymbol{\beta})}{\partial K^i}\right] = \operatorname{sign}\left[V^p_{K^pK^i}(\boldsymbol{\beta})\right],\ i = 1,2,\ldots,P\,, \tag{25}$$

$$\operatorname{sign}\left[\frac{\partial I^{p*}(\boldsymbol{\beta})}{\partial q}\right] = \operatorname{sign}\left[V^p_{K^pq}(\boldsymbol{\beta}) - 1\right], \tag{26}$$

$$\operatorname{sign}\left[\frac{\partial I^{p*}(\boldsymbol{\beta})}{\partial w^{i-P-1}}\right] = \operatorname{sign}\left[V^p_{K^pw^{i-P-1}}(\boldsymbol{\beta})\right],\ i = P+2, P+3,\ldots,2P+1\,, \tag{27}$$

$$\operatorname{sign}\left[\frac{\partial I^{p*}(\boldsymbol{\beta})}{\partial r}\right] = \operatorname{sign}\left[V^p_{K^pr}(\boldsymbol{\beta})\right]. \tag{28}$$

Equations (25)–(28) provide a considerable generalization of Proposition 1(ii) of Long (2010, Chapter 5) and the comparative dynamics results of Jun and Vives (2004, p. 256), as they both rely upon strong functional form and symmetry assumptions and limit the game to two players, whereas the symmetry and positive semidefiniteness of $\mathbf{S}^p\left(\mathbf{I}^*(\boldsymbol{\beta})\right)$ relies only upon assumptions (A.1) through (A.5).

Equation (25) asserts that the effect of an increase in the *i*th firm's capital stock on the *p*th firm's rate of investment is the same as the effect of an increase in the *i*th firm's capital stock on the *p*th firm's current value shadow price of its own capital stock. Using the taxonomy in Definitions 2 and 3 of Long (2010, Chapter 5), Eq. (25) shows that value function substitutability in the capital stocks is equivalent to Markov control-state substitutability, and does so, in general, for capital accumulation games. Thus Eq. (25) answers the call by Figuières (2009, p. 60) " … to explore the importance of complementarity and substitutability concepts in a class of dynamic games that goes beyond the linear quadratic specification." In addition, Eq. (25) shows that concavity of the *p*th firm's current value function in its own capital stock is equivalent to the *p*th firm's investment rate being a locally and weakly decreasing function of its own capital stock.

Equation (26) shows that the law of demand, i.e., $\partial I^{p*}(\boldsymbol{\beta})/\partial q \le 0$, is not intrinsic to the capital accumulation game. Instead, Eq. (26) shows that the law of demand holds if and only if an increase in the price of investment causes the current value shadow price of capital to increase by less than the increase in the price of investment, i.e., if and only if $V^p_{K^p q}(\boldsymbol{\beta}) \le 1$. Equation (27) shows that the rate of investment by the *p*th firm decreases as the maintenance cost of capital of any firm increases, if and only if, the *p*th firm's current value shadow price of its own capital stock decreases with an increase in the maintenance cost. Equation (28) has a similar economic interpretation. Roughly speaking, Eqs. (25) through (28) show that a firm's current value shadow price of its own capital stock and its rate of investment respond in a qualitatively identical manner to a parameter change.

The symmetry of $\mathbf{S}^p\left(\mathbf{I}^*(\boldsymbol{\beta})\right)$ yields the fundamental reciprocity conditions of the game. As an example, consider the implied reciprocity condition

$$V^p_{K^p w^\ell}(\boldsymbol{\beta})\frac{\partial I^{p*}(\boldsymbol{\beta})}{\partial q} = [V^p_{K^p q}(\boldsymbol{\beta}) - 1]\frac{\partial I^{p*}(\boldsymbol{\beta})}{\partial w^\ell},\ \ell = 1,2,\ldots,P\ . \tag{29}$$

Given the aforesaid form of $\mathbf{S}^p\left(\mathbf{I}^*(\boldsymbol{\beta})\right)$, it is not surprising that the reciprocity conditions also involve more than the individual partial derivatives of the investment demand functions with respect to the capital stocks and prices.

Because the strategic comparative dynamics effects vanish in the capital accumulation game, as noted above, the form of $\mathbf{S}^p\left(\mathbf{I}^*(\boldsymbol{\beta})\right)$ is identical to that of the prototype adjustment cost model of the firm. In other words, the basic qualitative comparative dynamics of the adjustment cost model of the firm and the above capital accumulation game are identical, in spite of the fact that the equilibria of the two models differ, in general. This begs the question: How might one then empirically test whether the capital accumulating firms under investigation behave competitively or strategically? Seeing as the qualitative comparative dynamics are identical in form in either case, it seems that the answer is beyond the scope of the current results. This conclusion, however, is not correct. In the competitive case, each firm's optimal investment rate is a function of the common discount rate and unit cost of investment, as well as its own maintenance cost of capital and capital stock. In the strategic case, each firm's optimal investment rate is likewise a function of the common discount rate and unit cost of investment, but instead is also a function of the maintenance costs and capital stocks of all the other firms, in addition to its own. This difference is empirically testable and indicates one way to test for competitive or strategic behavior on the part of the firms.

The next section contemplates a well-known sticky price differential game. It too satisfies the separability conditions of Corollary 1, but a seemingly simple change to its structure implies that Corollary 1 can no longer be applied, in which case Theorem 1 is fruitfully put to use.

5. A Sticky-Price Game

In this section Theorem 1 and Corollary 1 are applied to the sticky-price differential game of Fershtman and Kamien (1987), which is given by

$$V^{\iota}(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{q^{\iota}(\cdot)} \int_{t}^{+\infty} \left[p(\tau)q^{\iota}(\tau) - cq^{\iota}(\tau) - \tfrac{1}{2}[q^{\iota}(\tau)]^2 \right] e^{-r[\tau - t]} d\tau \tag{30}$$

$$\text{s.t. } \dot{p}(\tau) = s\left[a - q^{1}(\tau) - q^{2}(\tau) - p(\tau) \right],\ p(t) = p_t\,,$$

for $\iota \in \{1,2\}$, where $p(\tau)$ is the sticky price of the homogeneous good produced by the firms at time τ, $q^{\iota}(\tau)$ is the rate of output of the homogeneous good produced by firm ι at time τ, $c > 0$ is a parameter common to each firm's cost function, $r > 0$ is the common rate of discount, $a > 0$ is the reservation price of the good, $s > 0$ is the speed at which the sticky price adjusts to its value on the inverse demand curve, and $\boldsymbol{\beta} \stackrel{\text{def}}{=} (p_t, a, c, s, r)$. One may directly verify that assumptions (A.1) and (A.2) hold for this game. What's more, the results in Fershtman and Kamien (1987) show that assumptions (A.3)–(A.5) hold too.

The sticky-price game defined by Eq. (30) satisfies the separability conditions of Corollary 1, as is readily deduced by inspection. More generally, the preceding conclusion follows from the fact that the sticky-price game is a member of the linear-quadratic class of differential games. Therefore, by Corollary 1, the strategic comparative dynamics effects are identically equal to zero, just as they are in the capital accumulation game.

An application of Theorem 1 to the sticky-price game leads to a simple form for its 5×5 symmetric and positive semidefinite matrix $\mathbf{S}^{\iota}\left(\mathbf{q}^{*}(\boldsymbol{\beta})\right)$, $\iota\in\{1,2\}$, scilicet,

$$\begin{bmatrix} \left[1-sV_{pp}^{\iota}\right]\frac{\partial q^{\iota*}}{\partial p} & \left[1-sV_{pp}^{\iota}\right]\frac{\partial q^{\iota*}}{\partial a} & \left[1-sV_{pp}^{\iota}\right]\frac{\partial q^{\iota*}}{\partial c} & \left[1-sV_{pp}^{\iota}\right]\frac{\partial q^{\iota*}}{\partial s} & \left[1-sV_{pp}^{\iota}\right]\frac{\partial q^{\iota*}}{\partial r} \\ -sV_{pa}^{\iota}\frac{\partial q^{\iota*}}{\partial p} & -sV_{pa}^{\iota}\frac{\partial q^{\iota*}}{\partial a} & -sV_{pa}^{\iota}\frac{\partial q^{\iota*}}{\partial c} & -sV_{pa}^{\iota}\frac{\partial q^{\iota*}}{\partial s} & -sV_{pa}^{\iota}\frac{\partial q^{\iota*}}{\partial r} \\ -\left[1+sV_{pc}^{\iota}\right]\frac{\partial q^{\iota*}}{\partial p} & -\left[1+sV_{pc}^{\iota}\right]\frac{\partial q^{\iota*}}{\partial a} & -\left[1+sV_{pc}^{\iota}\right]\frac{\partial q^{\iota*}}{\partial c} & -\left[1+sV_{pc}^{\iota}\right]\frac{\partial q^{\iota*}}{\partial s} & -\left[1+sV_{pc}^{\iota}\right]\frac{\partial q^{\iota*}}{\partial r} \\ -\left[V_{p}^{\iota}+sV_{ps}^{\iota}\right]\frac{\partial q^{\iota*}}{\partial p} & -\left[V_{p}^{\iota}+sV_{ps}^{\iota}\right]\frac{\partial q^{\iota*}}{\partial a} & -\left[V_{p}^{\iota}+sV_{ps}^{\iota}\right]\frac{\partial q^{\iota*}}{\partial c} & -\left[V_{p}^{\iota}+sV_{ps}^{\iota}\right]\frac{\partial q^{\iota*}}{\partial s} & -\left[V_{p}^{\iota}+sV_{ps}^{\iota}\right]\frac{\partial q^{\iota*}}{\partial r} \\ -sV_{pr}^{\iota}\frac{\partial q^{\iota*}}{\partial p} & -sV_{pr}^{\iota}\frac{\partial q^{\iota*}}{\partial a} & -sV_{pr}^{\iota}\frac{\partial q^{\iota*}}{\partial c} & -sV_{pr}^{\iota}\frac{\partial q^{\iota*}}{\partial s} & -sV_{pr}^{\iota}\frac{\partial q^{\iota*}}{\partial r} \end{bmatrix}, \quad (31)$$

where $\mathbf{q}^{*}(\boldsymbol{\beta})$ is the feedback Nash equilibrium of the game. Furthermore, it follows from Theorem 1 that $\operatorname{rank}\left(\mathbf{S}^{\iota}\left(\mathbf{q}^{*}(\boldsymbol{\beta})\right)\right)\le\min(M^{\iota},N+K+1)=\min(1,1+3+1)=1$. Note that much of the discussion surrounding the corresponding matrix for the capital accumulation game applies to Eq. (31), and thus will not be repeated. Instead, the focus in what follows is on the additional information forthcoming because of the linear-quadratic structure of the game.

Equations (3.2) and (3.8) of Fershtman and Kamien (1987) show that $V^{\iota}(\cdot)$ is convex in p for both firms when a stable, symmetric, and linear feedback Nash equilibrium is considered, i.e., $V_{pp}^{\iota}(\boldsymbol{\beta})\ge 0$ for $\iota\in\{1,2\}$ in this instance. They also demonstrated that $1-sV_{pp}^{\iota}(\boldsymbol{\beta})>0$ holds along the aforesaid equilibrium path. It thus follows from Eq. (31) that $\partial q^{\iota*}(\boldsymbol{\beta})/\partial p\ge 0$ for $\iota\in\{1,2\}$, which recovers Corollary 2 in Fershtman and Kamien (1987). In other words, the law of supply holds for the stable, symmetric, and linear feedback Nash equilibrium. It should be emphasized, however, that the latter result is forthcoming only because it was possible to derive a closed-form solution for the feedback Nash equilibrium, which itself is due to the linear-quadratic structure of the sticky-price game. As is now shown, even a simple change to the structure of the game can profoundly change the structure of its intrinsic comparative dynamics.

Lambertini (2010) modified the sticky-price game of Fershtman and Kamien (1987) by replacing the inverse linear demand function with an inverse hyperbolic demand function. This caused the state equation to change to $\dot{p}(\tau) = s\left[\left[a/[q^1(\tau)+q^2(\tau)]\right] - p(\tau)\right]$, but left the game otherwise unaffected. This seemingly simple change has two implications for the present results. First, it is no longer possible to derive a closed-form solution for a symmetric feedback Nash equilibrium, as remarked by Lambertini (2010, p. 109). Consequently, many of the results derived by Fershtman and Kamien (1987) need not hold in Lambertini's (2010) version of the game. Second, the separability assumptions in Corollary 1 no longer hold for the transition function, so that Corollary 1 no longer applies to the modified game. As a result, the strategic comparative dynamics effects are part of the intrinsic comparative dynamics in this case.

To see the latter point, let $\iota = 1$ and consider, e.g., the (1,1) element $S^1_{pp}\left(\mathbf{q}^*(\boldsymbol{\beta})\right)$ of $\mathbf{S}^1\left(\mathbf{q}^*(\boldsymbol{\beta})\right)$ from Eq. (31). Given that $g(p,q^1,q^2;a,s) \overset{\text{def}}{=} s\left[\left[a/[q^1+q^2]\right] - p\right]$, it follows from Theorem 1 that

$$\begin{aligned} S^1_{pp}\left(\mathbf{q}^*(\boldsymbol{\beta})\right) &= \left[1 + g_{q^1} V^1_{pp}\right]\frac{\partial q^{1*}}{\partial p} + V^1_p g_{q^1q^2}\frac{\partial q^{1*}}{\partial p}\frac{\partial q^{2*}}{\partial p} \\ &= \left[\left[1 + g_{q^1} V^1_{pp}\right] + V^1_p g_{q^1q^2}\frac{\partial q^{2*}}{\partial p}\right]\frac{\partial q^{1*}}{\partial p} \geq 0. \end{aligned} \tag{32}$$

The second term in the first line on the right-hand side of Eq. (32) is the strategic comparative dynamics effect, which heretofore has been absent in the two contemplated games. As remarked above, it is not possible to derive a closed-form solution for a symmetric feedback Nash equilibrium when the inverse demand function is hyperbolic. When this is combined with the facts that $g_{q^1}(p,q^1,q^2;a,s) = -s\,a/[q^1+q^2]^2 < 0$ and $g_{q^1q^2}(p,q^1,q^2;a,s) = 2sa/[q^1+q^2]^3 > 0$, it implies that $\partial q^{1*}(\boldsymbol{\beta})/\partial p$ may be either positive or negative. In other words, the law of supply may not hold because of the change from a linear, to a hyperbolic, inverse demand function. Consequently, the law of supply is not intrinsic to the sticky-price differential game, as it is contingent upon the assumption of a linear inverse demand function. What is intrinsic to the sticky-price game, in general, that is, with a general form of a downward sloping inverse demand function and general cost function, is the symmetry and positive semidefiniteness of $\mathbf{S}^{\iota}\left(\mathbf{q}^{*}(\boldsymbol{\beta})\right)$, $\iota \in \{1,2\}$, as given by Theorem 1. The detailed form of $\mathbf{S}^{\iota}\left(\mathbf{q}^{*}(\boldsymbol{\beta})\right)$ in this case is left for the reader to consider.

6. Concluding Remarks

A complete characterization of the comparative dynamics of a locally differentiable feedback Nash equilibrium for the ubiquitous class of autonomous, exponentially discounted, and infinite horizon differential games has been achieved. A symmetric and positive semidefinite matrix has been show to provide the said characterization, and an upper bound to the rank of the matrix has also been derived. Assumptions that transcend those required for a locally differentiable characterization of a differential game's comparative dynamics were not made, thereby implying that the results given in Theorem 1 are basic, fundamental, or intrinsic to, all sufficiently smooth feedback Nash equilibria of the aforesaid class of differential games. These results are heretofore unknown properties of feedback Nash equilibria of the contemplated class of differential games, and constitute their basic, empirically testable properties.

The form of the intrinsic comparative dynamics consists of a nonstrategic component, viz., a generalized Slutsky-like expression, and a strategic component which captures of the effects that the other $P-1$ players have on a given player's basic comparative dynamics. Sufficient conditions in the form of easy-to-verify additive separability conditions on the instantaneous payoff and transition functions with respect to the control variables were identified that render the strategic comparative dynamics component identically zero. It turns out that many of the applied differential games meet the separability conditions, thus indicating the applicability and utility of Corollary 1 in applied modeling. Moreover, for differential games that meet the separability conditions, their intrinsic comparative dynamics are identical in form to those in optimal control problems, thereby demonstrating that even though such differential games are inherently strategic, their intrinsic comparative dynamics are devoid of strategic effects.

7. References

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